A Comprehensive Review of Solving Selective Harmonic Elimination Problem With Algebraic Algorithms

Chenxu Wang, Qi Zhang, *Student Member, IEEE*, Wensheng Yu, and Kehu Yang, *Member, IEEE*

***Abstract*—Selective harmonic elimination pulsewidth modulation (SHEPWM) is an effective way to eliminate low-order harmonics in high-power applications. However, one of the biggest challenges of SHEPWM is to solve the selective harmonic elimination (SHE) equations, which are composed of some nonlinear transcendental equations. Over the past few decades, algebraic algorithms have shown a considerable ability to solve SHE equations, specifically for obtaining all exact solutions. Much research has been published about algebraic algorithms, struggling to solve more switching angles, solving different mathematic models of SHEPWM, and so on. This article comprehensively reviews existing algebraic algorithms, including elementary symmetric polynomials, power sums, Newton's identities, resultant elimination method, Wu's method, Gröbner-basis-based method, Chudnovsky algorithm, polynomial homotopy continuation algorithm, and real-time implementation by algebraic algorithms. The principle operation of these methods is summarized, and their performance is analyzed in terms of execution time, solving ability, and applicability for different mathematical models.**

***Index Terms*—Algebraic algorithms, dc–ac conversion, high-power applications, inverters, renewable energy system, selective harmonic elimination (SHE).**

I. Introduction

Recently, because of the rapid development of renewable energy generation and high-power motor-driven applications, high-power converters have played a significant role in our daily lives [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14]. However, high-power converters bring great challenges to energy conversion technology, especially to the modulation technology. High-power converters commonly operate at high voltage and current, leading to increased switching losses in power semiconductor devices [15], [16], [17], [18], [19]. Furthermore, the high voltage and current also lead to challenges related to electromagnetic interference (EMI), parasitic capacitances, and voltage spikes. Therefore, high-power converters have to utilize low switching frequency to reduce power losses and to avoid EMI issues. However, the low switching frequency can exacerbate harmonic distortion in the output waveform. The harmonics, especially the low-order harmonics, are harmful to the power system elements, and minimizing harmonics for converters is crucial to comply with grid standards and prevent damage to sensitive connected devices [20], [21], [22], [23], [24], [25], [26], [27]. Thus, the challenge of dealing with low switching frequency and low-order harmonics is a significant concern, and engineers must balance these seemingly contradictory aspects when formulating high-power converter designs.

Among the commonly used pulsewidth modulation (PWM) technologies, the selective harmonic elimination (SHE) PWM stands out as a remarkably effective method for mitigating low-order harmonics, particularly when operating at exceedingly low switching frequencies [28], [29], [30], [31], [32], [33], [34]. Based on the principles of Fourier series expansion, selective harmonic elimination pulsewidth modulation (SHEPWM) employs a mathematical model to represent the harmonic constituents of a PWM waveform through a set of nonlinear and transcendental equations, known as SHE equations [35], [36]. These equations dictate the switching angles necessary for generating PWM drive signals. It is valuable to note that solving SHE equations poses significant challenges due to their inherent nonlinear and transcendental characteristics. Moreover, most of the SHE equations yield multiple solutions that are often discontinuous [37], [38]. As a result, obtaining all solutions for SHE equations stands as a substantial hurdle within the SHEPWM strategy. Addressing this challenge ranks among the foremost concerns associated with the practical implementation of SHEPWM.

In recent years, many algorithms have been proposed to solve the SHE equations, which can be classified as

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Chenxu Wang is with the School of Mechanical and Electrical Engineering, China University of Mining and Technology (Beijing), Beijing 100083, China (e-mail: eechenxu@163.com).

Qi Zhang is with the AAU Energy, Aalborg University, DK-9220 Aalborg, Denmark (e-mail: qzg@energy.aau.dk).

Wensheng Yu is with the School of Electronics Engineering, Beijing University of Posts and Telecommunications, Beijing 100876, China (e-mail: wsyu@bupt.edu.cn).

Kehu Yang is with the School of Artificial Intelligence, China University of Mining and Technology (Beijing), Beijing 100083, China, and also with the Key Laboratory of Intelligent Mining and Robotics, Ministry of Emergency Management of the People's Republic of China, Inner Mongolia Research Institute of China University of Mining and Technology (Beijing), Ordos 017004, China (e-mail: ykh@cumtb.edu.cn).

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numerical, bioinspired intelligent, and algebraic algorithms. Numerical algorithms can quickly obtain solutions because of the fast convergence speed, but they highly rely on the guess of initial values [39], [40], [41], [42], [43], [44], and it cannot solve all solutions of SHE equations. Bioinspired intelligent algorithms, such as genetic algorithm, particle swarm optimization algorithm, and differential evolution algorithm, have been well developed in recent years [45], [46], [47], [48], [49], [50], [51], [52], [53], [54], [55], [56], [57]. Although bioinspired intelligent algorithms can randomly choose initial values, but they lack the support of the mathematical theory, and they are sensitive to input parameters, so they are easy to fall into an optimal local solution and cannot deal with the multisolution feature of SHE equations. Furthermore, neither numerical algorithms nor bioinspired intelligent algorithms cannot give a clear conclusion on whether SHE equations have solutions or not under a certain modulation index. When these algorithms fail to provide a final result, the reason for the failure cannot be given by the algorithm itself, maybe it is caused by the selection of initial values, or maybe the parameters of the algorithm are unsuitable, or maybe there are indeed no solutions for the SHE equations.

The algebraic method has developed rapidly in recent years in solving SHE equations because of its outstanding advantages.

1) They can directly solve SHE equations without any initial value.
2) All solutions under full modulation index can be obtained.
3) The solutions solved by algebraic algorithms are precise.
4) They can give explicit conclusion about whether the SHE equations have a solution or not.

For example, the resultant elimination method [58], the Wu method [59], the Gröbner-basis-theory-based method [37], and the polynomial homotopy continuation method [60] are reported to be used for solving SHE equations. These methods have verified the powerful ability and outstanding advantages of algebraic methods in solving SHE equations. Besides, with the development of computer algebra technology in recent years, more and more algebraic methods may be used to solve SHE equations. Moreover, algebraic methods also show powerful capabilities in the simplification of SHE equations, such as the elementary symmetric polynomial method [61] and Newton's identities method [62]. With those methods, not only the solving efficiency can be improved but also the maximum solvable number of switching angles can be increased. Furthermore, it has been reported that some algebraic methods [63] can be parallelized to reduce the time consumption. It is also valuable to be pointed out that algebraic algorithms can be used for real-time implementation of SHE modulation [64].

Consequently, it can be said that the algebraic-algorithm-based method has gradually formed a new research area for solving SHE equations. To give researchers and engineers a clear and better understanding on the algebraic algorithms, this article provides a comprehensive review of algebraic algorithms, including the limitations and preconditions of these algorithms for different SHE models and the algorithm's complexity and efficiency. This single article will be helpful to researchers and engineers who are working to improve the efficiency and performance of inverters.

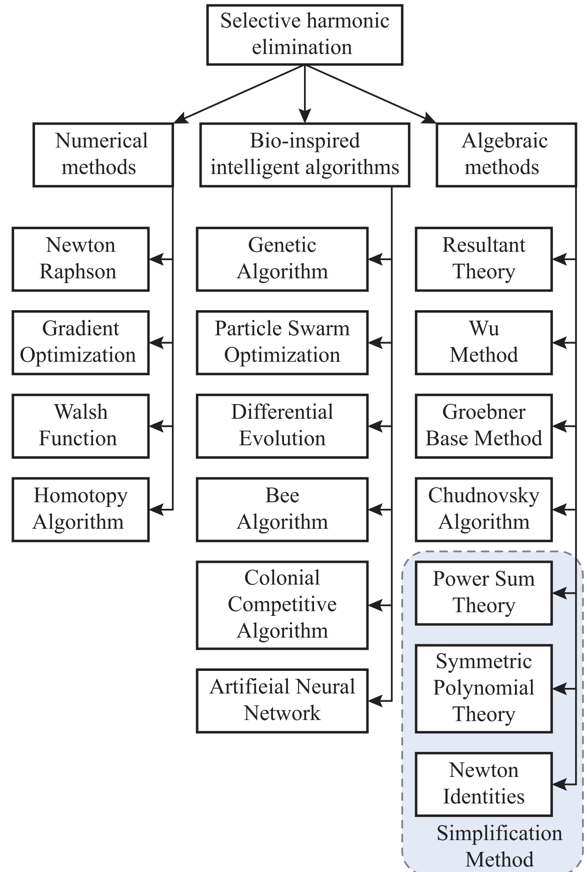

Fig. 1. Classification of SHEPWM solving methods.

The rest of this article is organized as follows. Section II summarizes and overviews the existing solving method of SHE equations. Then, in Section III, the main steps to solve SHE equations using algebraic algorithm are concluded. Section IV focuses on the simplification of SHE equations with the algebraic algorithm, and Section V summarizes the principle and solving process of the commonly used algebraic algorithms. In Section VI, the real-time implementation methods of algebraic algorithms are summarized. In Section VII, the performance and comparisons of these methods are presented. In Section VIII, the experiential results of the SHEPWM based on algebraic methods are given. Finally, Section IX concludes this article.

II. Overview of SHEPWM Solving Methods

To adopt the SHEPWM strategy for converters, SHE equations must be solved to generate a PWM drive signal. The reported solving methods in the literature can be divided into three families, namely, the numerical methods, the bioinspired intelligent methods, and the algebraic methods, as shown in Fig. 1. In this section, those methods are summarized and compared in a high viewpoint, with the advantages and limitations.

A. Numerical Methods

As one of the most traditional approaches, numerical methods employ iterative algorithms to compute solutions for SHE equations. Numerical methods are favored due to their ability to quickly provide precise and accurate solutions for SHEPWM. However, their effectiveness heavily depends on the selection of initial values. For many SHE equations, there is no universal method to choose these initial values, especially for multilevel converters. Typically, initial values required for numerical methods are determined through trial and error, often relying on experiential knowledge. Several predictive methods [40], [65] have been proposed to estimate initial values for numerical methods, but their capabilities are limited, and they can increase the complexity of the solution process. Moreover, numerical methods tend to converge to local minimum solutions, thus often providing only one feasible solution for SHE equations. Obtaining all possible solutions of SHE equations through numerical methods is impossible. Recent literature has introduced improved numerical methods such as Walsh functions [39], [66], [67], [68], homotopy algorithms [60], [69], [70], [71], and gradient optimization [72]. While these methods have enhanced performance to some extent, the challenge of accurately guessing initial values remains a significant obstacle.

B. Bioinspired Intelligent Algorithms

Bioinspired intelligent algorithms [48], [73], [74], [75], [76], [77], [78], [79], [80] are a category of algorithms that draw inspiration from natural laws and phenomena, such as species migration, natural selection, ant colonies, human culture, honeybee colonies, swarms of birds, and schools of fish. Compared to numerical methods, bioinspired intelligent algorithms stand out due to their minimal requirements for initial values. The initial values for these algorithms have only a modest impact on the search for optimized solutions, allowing them to be randomly selected or guessed. In contrast to initial values, objective functions play a pivotal role in influencing the performance of bioinspired intelligent algorithms. These objective functions are typically designed by researchers and engineers based on their own experiences, and as a result, the performance of these algorithms can be influenced by the designer's expertise. Furthermore, bioinspired intelligent algorithms introduce many parameters, such as population size, activation functions, weighting factors, allocation factors, etc. Achieving high performance with these algorithms requires careful parameterization. However, parameterization is not a straightforward task because: 1) there is no universally accepted method to determine these parameters and 2) there are numerous parameters, and they are interrelated. Thus, coordinating these parameters becomes essential for optimal performance. In summary, bioinspired intelligent algorithms offer a unique approach to problem solving, characterized by their flexibility in handling initial values and their dependence on well-designed objective functions and carefully tuned parameters for achieving peak performance.

C. Algebraic Algorithms

Algebraic algorithms [35], [38], [81], [82], [83], [84] are based on symbolic computation, in which all variables are represented with their symbols, not specific values, either during a single iteration or over the entire computational cycle. Algebraic algorithms first transform SHE equations into a high-order polynomial system. Afterward, some symbolic computation theories can be applied to solve SHE equations. Using symbols to represent variables is outstanding because of its unrivaled accuracy, including the precision of every solution and the capability to obtain all solutions of a nonlinear equation. Compared with numerical and bioinspired intelligence algorithms, algebraic algorithms have a strong ability to provide all feasible solutions of SHE equations with only one solving procedure, without requiring any initial values. Obtaining all solutions of SHE equations is crucial for achieving performance optimization in SHEPWM techniques, especially in high-power applications such as high-power motor drivers and high-power grid-connected inverters. Currently, the most commonly used algebraic algorithms for solving SHE equations include resultant theory, the Wu method, and Gröbner basis theory [37]. It is worth noting that power sum theory and symmetric polynomial theory are reported to simplify SHE equations, reducing their complexity. These simplification methods are decoupled from the main solving process, meaning that they can also be applied to numerical and bioinspired intelligence algorithms as preprocessing steps to reduce computational complexity. Algebraic algorithms have also been reported for use in real-time implementations of SHEPWM. In summary, algebraic algorithms provide a powerful tool for solving SHE equations. However, computational complexity and computational burden remain significant challenges. The following sections will provide detailed information on the principles, implementation, advantages, and limitations of algebraic algorithms.

III. Main Steps to Solve the SHE Problem With Algebraic Algorithms

A. Mathematic Model of SHE

The principle of SHEPWM strategy is the Fourier series decomposition of the period PWM voltage waveform generated by a power electronics converter, as given by

$$V(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\theta) + b_k \sin(k\theta)\right). \qquad (1)$$

According to the different dc voltages of converters, the symmetry feature of the output waveform, and the number of output levels, several mathematic models of SHE can be derived from (1) [85], [86], [87]. First, the SHE model can be divided into equal- and unequal-level models [88], [89], [90], [91] according to the dc voltage of converters. Second, the symmetry feature of the output waveform can determine quarter-symmetry, half-symmetry [92], [93], [94], and nonsymmetry SHE models. Third, based on the level of the output waveform, the SHE model can be classified as two-level [95], [96], [97] and multilevel [98], [99], [100], [101], [102], [103], [104]. The commonly used mathematic models and the corresponding output waveform have been summarized in Fig. 2. In these mathematic models, a_k represents the amplitude of the sine components of harmonic, and b_k represents the amplitude of the cosine components of

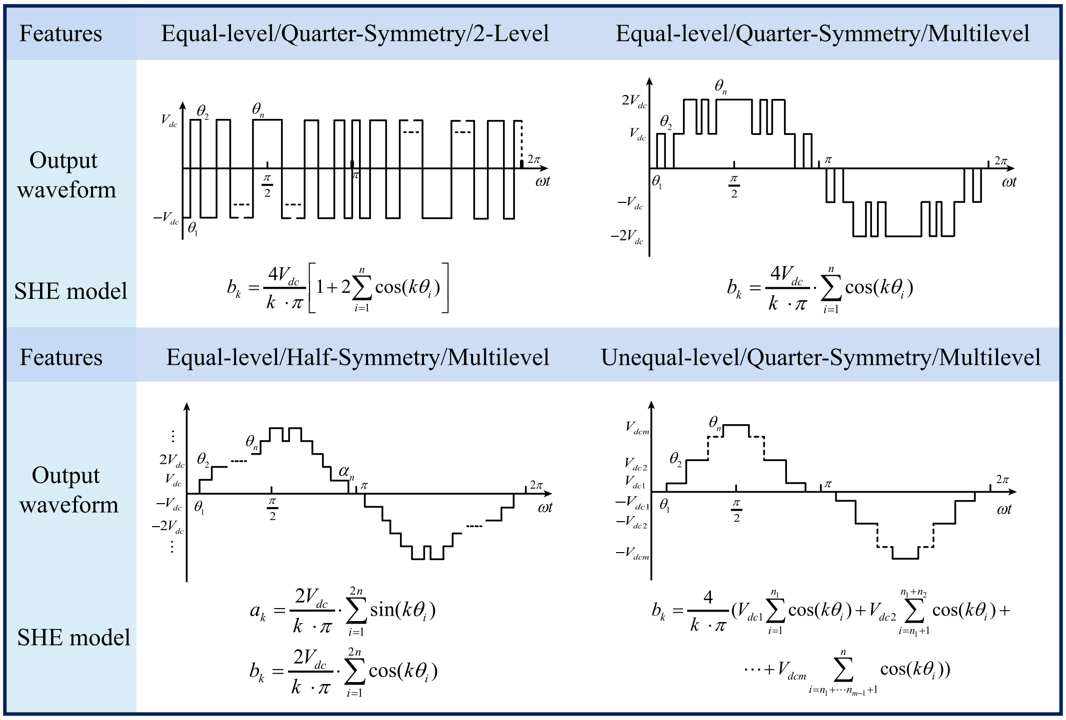

Fig. 2. Commonly used mathematic models of SHE.

harmonic. For the quarter-symmetry SHE models, the dc component, even harmonics, and sine components are automatically eliminated due to the symmetry, so only the cosine term needs to be controlled. For the half-symmetry models, the dc component and even harmonics are eliminated, so both the sine and cosine terms need to be controlled in the half-symmetry models. Besides, V_{dc} represents the value of dc power supply. In some actual industrial applications, the dc voltage of power supply is different and even time varying, so the unequal-level SHE models need to be used. θ is the switching angle, which is determined by two factors: one is the state of the edge, that is, rising or falling, and the other is the value of θ. These two factors of θ determine different SHE waveforms, commonly referred to as the switching pattern. Since only one switching pattern is possible for a two-level waveform, only one solution exists for the two-level SHE models, but usually multiple solutions exist for the multilevel SHE models [102], [103], [104]. No matter which SHE models are listed in Fig. 2, all switching patterns can be obtained within one solving process by using algebraic algorithms.

B. Procedure of Solving SHE Equations With Algebraic Algorithms

In this section, the relatively unified solving process for SHE equations is extracted from the literature on many algebraic algorithms so that readers can have an overall grasp of the solving process and make it easier to evaluate the algorithm's performance later. Generally, the standard procedure for solving SHE problems with algebraic algorithms contains three main steps, as shown in Fig. 3.

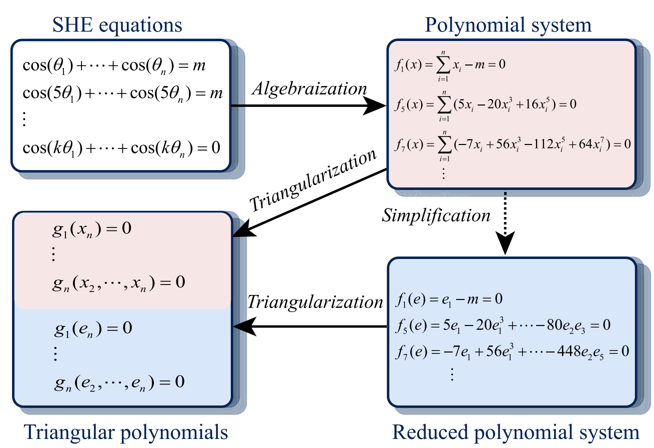

Fig. 3. Procedure of solving SHE problems by algebraic algorithms.

1) Step 1—The Algebraization of the Cosine Form SHE Equations: Taking the mathematic model of three-phase quarter-wave (QW)-symmetry multilevel SHE as example, the SHE equations can be expanded as

$$\begin{cases} \cos(\theta_1)+\cos(\theta_2)+\cos(\theta_3)+\cdots+\cos(\theta_{\mathrm{n}})=\mathrm{m} \\ \cos(5\theta_1)+\cos(5\theta_2)+\cos(5\theta_3)+\cdots+\cos(5\theta_n)=0 \\ \cos(7\theta_1)+\cos(7\theta_2)+\cos(7\theta_3)+\cdots+\cos(7\theta_n)=0 \\ \vdots \\ \cos(k\theta_1)+\cos(k\theta_2)+\cos(k\theta_3)+\cdots+\cos(k\theta_n)=0 \end{cases} \tag{2}$$

where k is the order of the eliminated harmonic; m is the modulation index, which defines the ratio of the fundamental amplitude b_1 and the dc-link voltage V_{dc}

$$m = \frac{\pi b_1}{4V_{\mathrm{dc}}}. \tag{3}$$

It can be seen that the SHE equations are all cosine functions in switching angles, which cannot be directly solved by algebraic algorithms. As the multiple angle cosine and the Chebyshev polynomial [105] have the following relationship, (2) can be converted to an algebraic polynomial system:

$$\cos(k\theta_i) = T_k(\cos(\theta_i)). \tag{4}$$

Let $x_i = \cos(\theta_i)$; then, all the cosine terms in (2) can be converted to polynomials by using the following iterations:

$$\begin{cases} T_1(x) = x, \;\; T_2(x) = 2x^2 - 1 \\ \vdots \\ T_k(x) = 2xT_{k-1}(x) - T_{k-2}(x) \end{cases}. \tag{5}$$

Then, the original cosine-form SHE equations are converted to the following polynomial equations:

$$\begin{cases} f_1(x) = \sum_{i=1}^{n} x_i - m = 0 \\ f_5(x) = \sum_{i=1}^{n} \left(5x_i - 20x_i^3 + 16x_i^5\right) = 0 \\ f_7(x) = \sum_{i=1}^{n} \left(-7x_i + 56x_i^3 - 112x_i^5 + 64x_i^7\right) = 0 \\ \vdots \end{cases} \cdot \tag{6}$$

It can be seen that the algebraic-form SHE equations (6) are with multivariables and a high degree of these variables. The algebraic algorithms can directly solve (6), but the computational complexity is exceptionally high. Generally, the degree of (6) can be reduced before using the algebraic algorithm. More than one simplification method has been proposed in recent years [62], [83], [106]. The elementary symmetric polynomial is the most commonly used in the existent simplification methods [81], [82], [83], so take it as an example here.

2) Step 2—The Simplification of the Algebraic Polynomial System: Based on the symmetry feature of (6), its degrees can be reduced by substituting the variables x_i with the elementary symmetric polynomials e_i. Then, the reduced polynomial system can be obtained as follows:

$$\begin{cases} f_1(e) = e_1 - m = 0 \\ f_5(e) = 5e_1 - 20e_1^3 + 16e_1^5 + \cdots - 80e_2e_3 = 0 \\ f_7(e) = -7e_1 + 56e_1^3 - 112e_1^5 + \cdots - 448e_2e_5 = 0 \\ \vdots \end{cases} \tag{7}$$

where e_1–e_n have an equivalent relation with x_1–x_n, called the elementary symmetric polynomials, and it will be detailed described in next section.

As the modulation index m is known, by substituting $e_1 = m$ into the following equations, it can be seen that the degrees of the unsolved variables are greatly reduced, which dramatically accelerates the solving procedure.

3) Step 3—The Triangularization of the Reduced Polynomial System: Finally, the algebraic algorithms can be applied to solve the reduced polynomial system (7). For example, if the Gröbner-basis-based method [37] is used, the following form can be obtained, which are composed of a high-order univariate equation plus a set of univariate linear equations

$$\begin{cases} g_1(e_n) = a_m e_n^m + a_{m-1}e_n^{m-1} + \cdots + a_1 e_n = 0 \\ g_2(e_{n-1}, e_n) = b_1 e_{n-1} + f_1(e_n) = 0 \\ g_3(e_{n-2}, e_n) = b_2 e_{n-2} + f_2(e_n) = 0 \\ \vdots \\ g_{n-1}(e_2, e_n) = b_{n-2}e_2 + f_{n-2}(e_n) = 0 \end{cases} \tag{8}$$

where if the modulation index is preset, $a_m, \ldots, a_1$ and $b_1, \ldots, b_{n-2}$ are integers, and $f_1, f_2, \ldots, f_{n-2}$ are univariate polynomials in e_n.

Once the SHE equations are converted to the form of (8), the subsequent solving procedure is relatively straightforward. One thing that should be noted is that the exact form of (8) is different according to the algebraic algorithm used. For example, if the resultant elimination method is used, (8) is the form of triangular algebraic polynomial.

IV. Simplification Methods

The polynomial system (6) is composed of symmetric polynomials, which means that when any two variables within a polynomial are exchanged, the polynomial remains unaltered. Based on this feature, several symmetric polynomial theories can be employed to simplify (6) into a reduced polynomial system, thereby improving the efficiency and ability of algebraic algorithms. This section summarizes the principles and properties of the existing simplification methods, including the elementary-symmetric-polynomial-based-method [83], power sums [106], and Newton's identities-based method [62].

A. Elementary Symmetric Polynomials

The definition of the elementary symmetric polynomials is given in (10), which describes the relation between the original variables x and the elementary symmetric polynomials e. The principle of elementary symmetric polynomials indicates that any symmetric polynomial can be represented as the form of elementary symmetric polynomials. Due to the symmetrical property of the algebraic polynomial system (6), it can be represented as the form of elementary symmetric polynomials with e as shown in (7), so that (7) is relatively simple to be solved due to the reduced degree. After getting the results of e, the original variables x can be solved by constructing a univariate high-order polynomial (9). It is not hard to see that the elementary symmetric polynomials e are the coefficients of a univariate high-order polynomial with roots $x_1, x_2, \ldots, x_n$

$$\begin{aligned} f(x) &= x^n - e_1 x^{n-1} + \cdots + (-1)^{n-1}e_{n-1}x_1 + (-1)^n e_n \\ &= (x - x_1)(x - x_2)\cdots(x - x_n) \end{aligned} \tag{9}$$

$$\begin{cases} e_1 = x_1 + x_2 + \cdots + x_n \\ e_2 = x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n \\ e_3 = x_1x_2x_3 + x_1x_2x_4 + \cdots + x_{n-2}x_{n-1}x_n \\ \vdots \\ e_n = x_1x_2\cdots x_n \end{cases}. \tag{10}$$

Then, (7) will be solved by the subsequent algebraic methods, and the original variables $x_1, x_2, \ldots, x_n$ can be solved from e by constructing the univariate high-order polynomial (9) with the known coefficients of $e_1, e_2, \ldots, e_n$. By using the elementary-symmetric-polynomial-based method, the degree of the polynomial system (6) is significantly reduced, and the computational ability of the subsequent algebraic algorithms is improved.

However, simplifying the polynomial system by this method is a nonlinear process, which has a considerable computation burden with the increasing switching angles. In the symbolic computing software *Mathematica*, this simplification process can be done by calling the ⟨*SymmetricReduction*⟩ command, and the maximum number of switching angles that can be solved by *Mathematica* is 8. Besides, this method requires that SHE equations be symmetric polynomials, which can only be used to simplify equal-level SHE models.

B. Power Sum and Newton's Identities

Except for the elementary-symmetric-polynomial-based method, (6) can also be transformed into the form of power sum symmetric polynomials. The definition of power sum symmetric polynomials can be described as

$$p_k = \sum_{i=1}^{n} x_i^k. \tag{11}$$

Obviously, (6) can be linearly transformed into the form of power sum symmetric polynomial as (12). However, it can be seen that (12) is undetermined as its variables are always more than the equations, so it cannot be solved directly

$$\begin{cases} f_1(p) = p_1 - m = 0 \\ f_5(p) = 5p_1 - 20p_3 + 16p_5 = 0 \\ f_7(p) = -7p_1 + 56p_3 - 112p_5 + 64p_7 = 0 \\ \vdots \end{cases} . \tag{12}$$

Newton's identities describe the relation between power sum symmetric polynomials and elementary symmetric polynomials. Based on the Newton's identities, a balance can be achieved between the number of equations and variables in (12), leading to its representation as follows [62]:

$$\begin{cases} p_1 = e_1 \\ p_2 = e_1 p_1 - 2e_2 \\ p_3 = e_1 p_2 - e_2 p_1 + 3e_3 \\ \vdots \\ p_n = e_1 p_{n-2} - e_2 p_{n-3} + \cdots + (-1)^{n-2}(n-1)e_{n-1} \\ \vdots \\ p_i = e_1 p_{i-1} + e_2 p_{i-2} + \cdots + (-1)^{n+1} e_n p_{i-n} \end{cases} . \tag{13}$$

Therefore, the undetermined polynomial system (12) can be transformed into a balanced polynomial system (14) based on the theory of Newton's identities. Compared with the original polynomial system (6), the degree of (14) is significantly reduced

$$\begin{cases} f_1(p) = p_1 - m = 0 \\ f_5(p) = -20p_3 + \frac{40}{3}p_1^2 p_3 - \cdots + \frac{40}{3}p_3 p_2 + 5p_1 = 0 \\ f_7(p) = 56p_3 - \frac{280}{3}p_1^2 p_3 + \cdots + \frac{112}{3}p_3 p_2^2 - 7p_1 = 0 \\ \vdots \end{cases} . \tag{14}$$

TABLE I
COMPARISON OF THE DEGREE OF $f_{13}(x)$, $f_{13}(e)$, AND $f_{13}(p)$

	$x_1/e_1/p_1$	$x_2/e_2/p_2$	$x_3/e_3/p_3$	$x_4/e_4/p_4$	$x_5/e_5/p_5$
$f_{13}(x)$	13	13	13	13	13
$f_{13}(e)$	0	6	4	3	2
$f_{13}(p)$	0	4	3	3	2

Similar to the elementary-symmetric-polynomial-based method, the original variables x can be solved by constructing the univariate polynomial (9), with coefficients e derived from the results of p through the application of (13). Both the elementary-symmetric-polynomial-based method and the Newton's identities-based method can reduce the degree of the SHE equations. The degree comparison among the original SHE equations (6) and the reduced SHE equations (7) and (14) is presented in Table I. In the case of SHE equations with five switching angles taken as an example, the highest degree of the original SHE equations is 13. When the elementary-symmetric-polynomial-based method is applied, the highest degree is reduced to 6. When the Newton-identity-based method is used, the highest degree can be reduced to 4. Besides, Newton-identity-based method has a relatively simple solving process, which only involves the multiplications and additions, so the computational burden is much less than the elementary-symmetric-polynomial-based method [62].

V. Algebraic Algorithms

To obtain the multiple solutions and switching patterns with high accuracy, many algebraic algorithms have been explored and improved to solve SHE equations during the past 20 years. One kind of algebraic algorithm is to solve polynomials based on computational symbolic mathematics, aiming to eliminate variables and get a triangular polynomial system like (8) based on different elimination principles, including the resultant elimination method [35], the Wu method [107], and the Gröbner-basis-based method [37]. Besides, there are other kinds of algebraic algorithm for solving SHE equations, including the polynomial homotopy continuation algorithm [60] and the Chudnovsky algorithm [108]. These methods can solve all the exact solutions without any initial value, but they rely on some mathematical transformation or numerical operation instead of the principle of eliminating variables in the solving process.

A. Resultant Elimination Method

The resultant elimination theory to solve SHE equations was first proposed by John Chiasson, which has been applied to both two-level and multilevel converters [83], [109]. After the SHE equations consisting of n switching angles are transformed into a polynomial system with n variables, the first step of the resultant elimination method is integrating the polynomial system into a form where only one variable is seen as the unknown and all other variables are seen as coefficients. Denote $f_1(x_1, x_2, \ldots, x_n)$ and $f_2(x_1, x_2, \ldots, x_n)$ as two multivariate polynomials of the polynomial system $F = \{f_1, f_2, \ldots, f_n\}$, and their maximum degree of x_n are k_1 and k_2, respectively; then, f_1 and f_2 can be

TABLE II
PROCEDURE AND DEVELOPMENT OF THE RESULTANT ELIMINATION METHOD

Ref.	Year	Objective of the study	Remarks
[111] [112]	2002	To precisely eliminate harmonics for multilevel H-Bridge converters with SHE by the proposed resultant elimination theory.	All exact solutions of SHE equations with three switching angles are given. The 5th and 7th harmonics are eliminated for a seven-level H-Bridge converter.
[109] [113]	2003	To consider the situation of unequal DC voltage for converters, apply the resultant elimination theory on unequal level converters, eliminate the selective harmonics for them.	This paper solves all exact solutions for unequal-level SHE equations by using the resultant elimination theory, precisely eliminates the 5th and 7th harmonics for the unequal-level H-Bridge converters with SHE.
[35]	2004	To propose a unified solution scheme for SHE, and solve all switching angles under full modulation index.	All switching angles for all possible switching patterns are solved in this paper. The solutions are used to control multi-level inverters, eliminating the 5th, 7th and 11th harmonics.
[61] [83]	2005	To reduce the degree of SHE equations by elementary-symmetric-polynomial-based simplification method and reduce the computational burden of the resultant elimination method.	The complexity of SHE equations is reduced by using the elementary symmetric polynomials method, and the resultant elimination method can solve 5 switching angles for the first time by combined with the simplification method.
[106]	2005	Proposing the power-sum-based method to reduce the complexity of SHE equations.	The degree of SHE equations is reduced by using the power sums-based method, and 5 switching angles can be solved by combining this method with the resultant elimination method.
[63]	2016	To improve the efficiency of solving the *Sylvester* matrix in the parallel way.	The polynomial interpolation principle is applied to get the determinant of the *Sylvester* matrix, improving the solving efficiency of the resultant elimination method.

rewritten as follows:

$$\begin{cases} f_1(x_1, x_2, \ldots, x_n) = 0 \\ f_2(x_1, x_2, \ldots, x_n) = 0 \end{cases}$$
$$\Rightarrow \begin{cases} \boldsymbol{A}(x_1, x_2, \ldots, x_{n-1}) \cdot \boldsymbol{x_n} = 0 \\ \boldsymbol{B}(x_1, x_2, \ldots, x_{n-1}) \cdot \boldsymbol{x_n} = 0 \end{cases} \quad (15)$$

where $\boldsymbol{x_n}$ is a vector consisting of monomials generated by $x_n, x_n^2, \ldots, x_n^k$, k is k_1 or k_2

$$\boldsymbol{A} = [a_1(x_1, x_2, \ldots, x_{n-1}), \ldots, a_{k1}(x_1, x_2, \ldots, x_{n-1})]$$
$$\boldsymbol{B} = [b_1(x_1, x_2, \ldots, x_{n-1}), \ldots, b_{k2}(x_1, x_2, \ldots, x_{n-1})]. \quad (16)$$

Then, the resultant elimination method point out the two polynomials have common zero only if the determinant of *Sylvester* matrix equals zero. The *Sylvester* matrix of f_1 and f_2 is defined as (17), where the size of the matrix is $(k_1 + k_2) \times (k_1 + k_2)$. Besides, the resultant of f_1 and f_2 is defined as (18)

$$S = \begin{bmatrix} \boldsymbol{A} & 0 & \cdots & 0 & \boldsymbol{B} & 0 & \cdots & 0 \\ 0 & \boldsymbol{A} & \cdots & 0 & 0 & \boldsymbol{B} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{A} & 0 & 0 & \cdots & \boldsymbol{B} \end{bmatrix} \quad (17)$$

$$\mathrm{Res}(f_1, f_2) = \det(S). \quad (18)$$

For the polynomial system F with n variables, it is necessary to select $n-1$ groups of binary polynomials and then solve $n-1$ *Sylvester* matrices to eliminate one variable. The procedure of the resultant elimination method continuously eliminates variables by constructing and solving the *Sylvester* matrix until there is only one variable left so that the polynomial system is transformed into a triangular set form. Therefore, the principle of the resultant elimination method is relatively easy, and the solving process of this algorithm is shown in Algorithm 1, and a summary of the application of this method is provided in Table II.

However, the nonzero terms in the S matrix are all symbolic polynomials, and the dimension of the S matrix increases rapidly along with the number of eliminated variables. Therefore, in the process of solving the determinant of the S matrix, the intermediate expression will swell in polynomial expansion, which leads to the exhaustion of the computer's physical memory.

Algorithm 1: Resultant Elimination Method.

1: Eliminating variable x_n by $x_n = m - x_1 - \cdots - x_{n-1}$, according to $f_1(x)$ in (6).
2: **while** $n = 1$ **do**
3: Choosing arbitrary every two polynomials in the remainder of (6);
4: Computing the Sylvester matrix of the two polynomials;
5: One variable can be eliminated, $n-2$ variables remaining;
6: **end while**
7: A triangular polynomial system can be obtained, solve the univariate higher polynomial with x_1 at first;
8: Solving the solution of $x_2, \ldots, x_n$ from the triangular polynomial system by taking the solved variables back.

To solve the problem of intermediate expression swell, Yang et al. [63] proposed the polynomial interpolation method to convert the symbolic operation to the numerical operation. The main idea of this method is to substitute all variables in the S matrix with integers or rational numbers and then obtain the final result through multivariate interpolation. Take a univariate polynomial $g(x)$ as an example to clarify the principle of the polynomial interpolation method at first. Denote $g(x) = h_m x^m + h_{m-1} x^{m-1} + \cdots + h_0$, and then, the coefficients of $g(x)$ can be easily determined by solving the linear equations (19)

$$\begin{bmatrix} x_0^0 & x_0^1 & \cdots & x_0^m \\ x_1^0 & x_1^1 & \cdots & x_1^m \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ x_m^0 & x_m^1 & \cdots & x_m^m \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_m \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{bmatrix} \quad (19)$$

where x_0–x_m are $m+1$ interpolation points, which can always guarantee the uniqueness of $g(x)$, and the x matrix is a Vandermonde matrix, which is defined as $\boldsymbol{V}$ here.

However, the S matrix is composed of multivariate polynomials. Taking (17) as an example to explain the determinant of the S matrix solved by the polynomial interpolation method, the whole solving process can be divided into four steps.

1) *Step 1:* The total degree of variables needs to be evaluated to determine the number of interpolation points. A simple way is to add all the degree of variables in each column or row, and then, the total number of interpolation points can be defined as (20), where $n-1$ is the number of variables in (17), and d_i is the degree of x_i. Also, there is another way to get the number of interpolation points; the detailed principle can refer to [63]

$$D = \prod_{i=1}^{n-1}(d_i+1), \quad 1 < m < n. \tag{20}$$

2) *Step 2:* Determine the d_i+1 interpolation points for each variable $x_i, (i=1,2,\ldots,n-1)$, and the corresponding Vandermonde matrix $\boldsymbol{V_i}$ can be generated by the interpolation points $x_i^0, x_i^1, \ldots, x_i^{di}$. Then, the vector $\boldsymbol{y}$ in (19) can be solved by calculating the determinant of the S matrix on each multivariate interpolation points.
3) *Step 3:* After the matrices $\boldsymbol{V}$ and $\boldsymbol{y}$ are known, the coefficient matrix $\boldsymbol{h}$ can be uniquely determined by solving (19); for the multivariate polynomials, $\boldsymbol{V}$ and $\boldsymbol{y}$ can be defined as

$$\begin{aligned}\boldsymbol{V} &= \boldsymbol{V_1} \otimes \boldsymbol{V_2} \otimes \cdots \otimes \boldsymbol{V_{n-1}}\\ \boldsymbol{y} &= [\boldsymbol{y_1}, \boldsymbol{y_2}, \ldots, \boldsymbol{y_D}]\end{aligned} \tag{21}$$

where $\otimes$ represents the Kronecker product of two matrices.
4) *Step 4:* Since the coefficient matrix of $\boldsymbol{h}$ is the successive Kronecker product of Vandermonde matrices, (19) can be solved by the extended n-dimensional Björck–Pereyr algorithm [110], which can be implemented on a parallel computing system. Once the coefficient matrix $\boldsymbol{h}$ is solved, the final resultant polynomial can be obtained as

$$\text{Res}(f_1, f_2) = \sum_{i=1}^{D} \boldsymbol{h_i} B_i \tag{22}$$

where B_i is a basis that consists of monomials generated by $x_1, x_2, \ldots, x_n$, which are arranged in ascending order in pure dictionary order.

Due to the mutual independence between the interpolation points, the calculation of the function values of the interpolation points can be easily distributed to multicore systems for parallel calculating [63]. Therefore, this algorithm can solve the problem of intermediate expression swell and improve the efficiency of the resultant elimination method by the parallel implementation.

B. Wu Method

The Wu method was proposed by Wu Wentsun in the 1970s and has been used in theorem machine proving, mathematical science, system science, and so on [114], [115], [116]. In 2005, the Wu method was first applied to solve SHE problems in engineering applications [59]. The Wu method's main idea is to transform the polynomial system (6) into a triangular characteristic set, and the solutions of the characteristic set are equal to the original polynomial system. The essence of the Wu method is the same as the resultant elimination method, which is transforming the problem of solving multivariate nonlinear equations into the problem of solving linear triangular equations. The main difference between the two methods is the principle and process of eliminating variables. According to the Wu method, the primary step is to solve the characteristic set, and the solving process of the characteristic set is briefly introduced as follows.

Given the polynomial system $F = \{f_1, f_2, \ldots, f_n\}$, it is grouped by class, i.e., by the maximum subscript of variables, and the same kind belongs to the same group. Then, in each group, a triangular group P is constructed by choosing a polynomial with the lowest power of the primary variable x_i. Usually, P is not an ascending set, and it can be restructured into an ascending set by calculating the remainder of polynomials in P to expand polynomials. Taking the SHE equations with three switching angles, the detailed calculation of the characteristic set is described follows.

First, selecting a basic set P_1 in a given polynomial group, then, calculating the remainder of P_1 for each polynomial in F_1, all the nonzero remainder is recorded as R_1.

Second, adding all the polynomials of R_1 to polynomial group F_1 can get a new polynomial group $F_2 = \{F_1, R_1\}$. Then, selecting a basic set P_2 from F_2, calculating the remainder of P_2 for each polynomial in F_2, all the nonzero remainder is recorded as R_2.

Third, repeat these steps inductively. After finite repetition of elimination steps, the polynomial group F_k and the basic set P_k are obtained, so that any polynomial in F_k has zero remainder of P_k. If P_k is a contradictory ascending set (a polynomial is a nonzero constant), polynomial group $F=0$ is unsolvable; if P_k is not a contradictory ascending set, then P_k is the characteristic set of F. The simple algorithm procedure is shown in Algorithm 2.

C. Gröbner-Basis-Based Method

The Gröbner basis theory was proposed by an Austrian mathematician Bruno Buchberger in 1965 [117] and first applied to solve SHE problems in 2015 [37]. Like the aforementioned methods, the Gröbner basis theory also provides nonnumerical solutions for nonlinear algebraic polynomial systems in the way of iterative variable elimination.

However, the Gröbner basis theory is much more complex than the resultant elimination method and the Wu method, so that some necessary principles will be indicated intuitively here.

I is an ideal of a polynomial system F, if I satisfies the following three conditions.

1) $0 \in I$.
2) If $f, g \in I$, then $f+g \in I$.
3) If $f \in I, h \in F$, then $hf \in I$.

Based on the Hilbert's basis theorem, I can be generated by finite number of polynomials $p = (p_1, p_2, \ldots, p_m) \subset F$, denoted

Algorithm 2: Wu Method.

1: Suppose polynomials system is F_0, the number of variables is n, $N = n$, and R_{N-n} is a nonzero polynomial.
2: **while** $R_{N-n} \neq 0$ **do**
3: Select the polynomial with the lowest order in F_{N-n} as the basic set P_{N-n}, and the remaining polynomials called F_{N-n+1};
4: Computing the remainder of F_{N-n+1} divided by P_{N-n}, and the remainder called R_{N-n+1};
5: **if** The variables of R_{N-n+1} are $n-1$ **then**
6: Go back to step 4
7: **else**
8: Go back to step 5
9: **end if**
10: $n = n - 1$
11: **end while**
12: If P_{N-n} is not a contradictory ascending set, P_{N-n} is the characteristic set of F_0.

as $I = p$, and p is called a basis of I. The Gröbner basis is a kind of basis with this nice property, and it always can be constructed by using some specific algorithms. A Gröbner basis for an ideal I is a finite set of polynomials $g = (g_1, g_2, \ldots, g_n)$, if

$$\langle LT(g_1), LT(g_2), \ldots, LT(g_n)\rangle = \langle LT(f) : f \in I\rangle \quad (23)$$

where LT denotes the leading term of a polynomial. However, the Gröbner basis of the algebraic polynomial system is not unique, which is decided by choice of the monomial order. To solve the SHE equations, the order must be under the pure lexicographical monomial so that the SHE equations can be transformed into a triangular form

$$\begin{cases} f_1(x_1, x_2, \ldots, x_n) = 0 \\ f_2(x_1, x_2, \ldots, x_n) = 0 \\ \vdots \\ f_n(x_1, x_2, \ldots, x_n) = 0 \end{cases} \Longrightarrow \begin{cases} g_1(x_n) = 0 \\ g_2(x_{n-1}, x_n) = 0 \\ \vdots \\ g_n(x_1, x_2 \ldots, x_n) = 0 \end{cases}. \quad (24)$$

Therefore, the core problem of this method is how to solve the Gröbner basis. It can be done using the famous Buchberger algorithm and some improved versions, and the detailed solving process is given in Algorithm 3. As the implementation of this algorithm requires solid knowledge of the polynomials and professional programming skills, it is unrealistic to develop the algorithm by engineers. Some commercial symbolic computing software, such as *Maple* and *Mathematica*, provide some functions to compute the Gröbner basis, and it is user-friendly for the engineers. The simple algorithm procedure is shown in Algorithm 3. Besides, the development of the Gröbner basis method in the SHE problems is summarized in Table III.

The essence of the three algorithms is to solve polynomial systems with the elimination method and then get a triangular polynomial system. However, the elimination principle is entirely different, resulting in different computational abilities and efficiency. Once the SHE equations have been converted to their equivalent triangular form like (8), there are univariate polynomials and several linear equations which any mathematic software can quickly solve.

Algorithm 3: Gröbner Basis.

1: Suppose polynomials system is $f_1, f_2, \ldots, f_m$ and the number of variables is m.
2: **while** $m \geq 1$ **do**
3: $I \leq f_1, f_2, \ldots, f_m$ represents the ideal generated by $f_1, f_2, \ldots, f_m$;
4: Find the lowest common multiplier of f_i and f_j $(i \neq j)$, that is $x_1^{r1} x_2^{r2} \cdots x_n^{rn}$;
5: Cancel out the highest terms of f_i and f_j $(i \neq j)$, that is $s(f_1, f_2) = \dfrac{x_1^{r1} x_2^{r2} \cdots x_n^{rn}}{LT(f_i)} f_i - \dfrac{x_1^{r1} x_2^{r2} \cdots x_n^{rn}}{LT(f_j)} f_j$;
6: Find the remainder of the polynomial $s(f_i, f_j)$ divided by the ideal I, that is $\dfrac{I}{s(f_i, f_j)}$.
7: **if** $\dfrac{1}{s(f_i, f_j)} \neq 0$ **then**
8: set $\dfrac{1}{s(f_i, f_j)} = f_{m+1}$, then get a new set $F = \{f_1, f_2, \ldots, f_m, f_{m+1}\}$, go back to 2, $m = m + 1$.
9: **else**
10: Go back to step 2, $m = m - 1$.
11: **end if**
12: **end while**

D. Chudnovsky Algorithm

In [108], [118], [119], and [120], a kind of algebraic method for solving SHE equations was proposed, and in [120], this method is named Chudnovsky algorithm. The main properties of the Chudnovsky algorithm different from the aforementioned algorithms are as follows. 1) the computing speed of this method is much faster than them; 2) this method can only be used to solve single-phase SHE equations, which means the eliminated harmonics must be all odd harmonics; and 3) this method is based on mathematic formula transformations rather than computational symbolic mathematics.

According to the prior knowledge introduced in Section III, SHE equations can be transformed into the form of power sums, as described in (12), which describes the power sum form of three-phase SHE equations. However, (12) is an undetermined polynomial system because the number of variables is more than the number of equations. Actually, the power sum form of single-phase SHE equations is balanced, and all odd power sums can be directly obtained. For example, the power sum form of single-phase SHE equations with three switching angles can be written as (25). Obviously, the value of p_1, p_3, and p_5 can be easily solved if the modulation index is given

$$\begin{cases} f_1(p) = p_1 = m \\ f_3(p) = 4p_3 - 3p_1 = 0 \\ f_5(p) = 16p_5 - 20p_3 + 5p_1 = 0 \end{cases}. \quad (25)$$

TABLE III
PROCEDURE AND DEVELOPMENT OF THE WU METHOD AND THE GRÖBNER-BASIS-BASED METHOD

Ref.	Year	Objective of the study	Remarks
[59]	2005	To apply the Wu method to solve SHE equations for a three-phase half-bridge voltage source inverter.	All exact solutions for SHE equations with can be solved by the Wu method, and the switching angles' linear expressions based on the solutions are gained by the fitting theory.
[107]	2007	To combine the elementary symmetric polynomials method with the Wu method and improve the solving ability and efficiency.	The complexity of SHE equations is reduced by the elementary symmetric polynomials method and the solving efficiency of the Wu method can be significantly increased.
[82]	2015	To discuss the real number of solutions for SHE converters by using the Gröbner-basis-based method.	The number of solutions for SHE equations can be explicitly analyzed by the Gröbner-basis-based method.
[37]	2015	To solve SHE equations by using Gröbner basis under the pure lexicographic monomial order.	All exact solutions of SHE equations with 5 switching angles are given by the Gröbner-basis-based method for the first time.
[81]	2016	To combine the elementary symmetric polynomials method with the Gröbner-basis-based method and improve the solving ability and efficiency of the Groebner basis.	The elementary symmetric polynomials based method is combined with Gröbner-basis-based method and all exact solutions of SHE equations with 9 switching angles are obtained for the first time.
[64]	2019	To implement SHE in real time with the exact solutions solved by the Gröbner- basis-based method.	This paper proposed a real-time implementation method based on Gröbner basis and Sturm's theorem, which can be realized in microcontroller within several milliseconds.

Algorithm 4: Chudnovsky Algorithm.

1: Solve all the odd power sums p_{2i-1} from (25);
2: Compute $v_{2i} = 0, v_{2i-1} = \dfrac{-2p_{2i-1}}{2i-1}$, and the detailed definition of v can refer to [120];
3: Compute $g_i(0 \leq i \leq 2n)$ from $g_0 = e^{v_0}, g_i = \frac{1}{i}\sum_{k=1}^{i} k v_k g_{i-k}$;
4: Construct $Toeplitz$ system $\begin{bmatrix} g_n & \cdots & g_1 \\ \vdots & \ddots & \vdots \\ g_{2n-1} & \cdots & g_n \end{bmatrix} \begin{bmatrix} \tilde{p}_1 \\ \vdots \\ \tilde{p}_n \end{bmatrix} = -\begin{bmatrix} g_{n+1} \\ \vdots \\ g_{2n} \end{bmatrix}$ and set $p_i = (-1)^i \tilde{p}_i$;
5: Solve p_i from the $Toeplitz$ system and construct the univariate polynomial (9) to get the final results x_i;
6: Solve the switching angles by using $\alpha_i = \arccos(x_i)$.

Based on the principle of power sums and Newton's identities, if all power sums p are known, the result of the original variables x can be solved by constructing the univariate high-order (9). The Chudnovsky algorithm can obtain $f(x)$ by a series of mathematical transformations with fast speed. The transformation process contains three important steps: the first step is getting the logarithmic derivative of $f(x)$, which can be finally expressed as

$$\frac{f'(x)}{f(x)} = \sum_{n=0}^{\infty} \frac{p_n}{x^{n+1}} = \frac{n}{x} + \sum_{n=1}^{\infty} \frac{p_n}{x^{n+1}} \tag{26}$$

and then, integrating (26) and $f(x)$ can be written as follows:

$$f(x) = x^n \exp\left(-\sum_{n=1}^{\infty} \frac{p_n}{nx^n}\right). \tag{27}$$

The second step is to construct $\frac{f(x)}{f(-x)}$ to eliminate all unknown even power sums p.

$$\frac{f(x)}{f(-x)} = (-1)^n \exp\left(-2\sum_{n=1,n\text{odd}}^{\infty} \frac{p_n}{nx^n}\right) \tag{28}$$

The right-hand side of (28) can be expanded based on Taylor series to obtain a polynomial with odd powers sums p as coefficients. All coefficients of $f(x)$ can be solved by matching the coefficients on both sides of (28). Therefore, the third step is to construct the matching equations, which can be expressed as

$$f(x) = (-1)^n f(-x) G(1/x) \tag{29}$$

where $G(x)$ is also a polynomial and its coefficients g_i can be solved from (28). Once $f(x)$ is determined, the results of x can be easily solved. The detailed solving process can refer to [108] and [120], and the deduced calculation formula has been roughly given in Algorithm 4.

The Chudnovsky algorithm efficiently solves the SHE equation by primarily employing mathematical formula conversions rather than extensive symbolic operations. Consequently, the Chudnovsky algorithm minimizes computational time and memory usage, expanding the range of solvable switching angles compared to the aforementioned algebraic methods. Besides, as shown in (25), the value of all odd power sums $p_1, p_3, p_5, \ldots$ must be calculated so that the subsequent functions can be constructed. However, in the three-phase SHE equations, the odd power sums cannot be directly obtained. As shown in the three-phase original algebraic equations (12), there are only $p_1, p_5, p_7, \ldots$ in the equations. Therefore, the Chudnovsky algorithm can only be used to solve the single-phase SHE equations.

E. Polynomial Homotopy Continuation Algorithm

Although the homotopy algorithm is a kind of numerical algorithm, it can overcome the local convergence characteristics of numerical methods [71]. When $F(x)$ is a polynomial system, the homotopy algorithm has its particular form, theory, algorithm, and software package and is closely related to the theory of algebraic geometry.

Denote $F(x) = \{f_1, f_2, \ldots, f_n\}$ as a polynomial system in which f_i are polynomials in variables $x_1, x_2, \ldots, x_n$. The polynomial homotopy continuation algorithm operates in two main stages. In the first stage, a start system called the homotopy mapping equation is constructed as

$$H(x,t) = c(1-t)F(x) + tG(x) = 0 \tag{30}$$

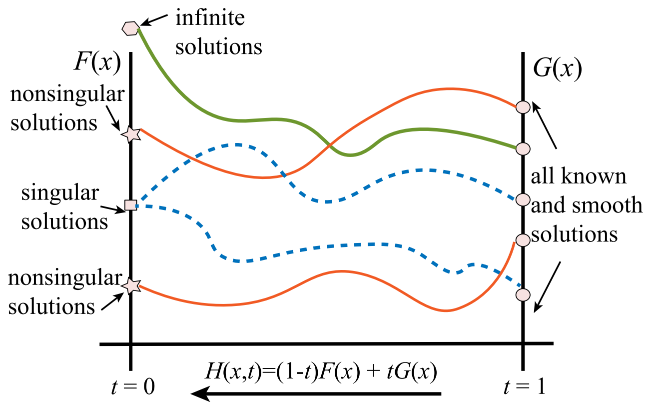

Fig. 4. Schematic diagram of the polynomial homotopy continuation algorithm.

where $G(x) = H(x, 0)$ is the start system and $F(x) = H(x, 1)$ is the target system that needs to be solved, c is a random complex number, and t is the homotopy parameter that is moving from 0 to 1. In order to find all solutions for $F(x)$, the solution count of the start system $G(x)$ cannot be less than $F(x)$. Denote E is the total degree of $F(x)$:

$$E = \prod_{i=1}^{n} deg(f_i) \tag{31}$$

where $deg(f_i)$ is the degree of the polynomial f_i of $F(x)$. Based on Bézout's theorem, if $F(x)$ has finite solutions in the complex field, then the number of the solutions is no more than E, so, the start system $G(x)$ should also have E solutions which originate E paths connect to the E solutions of $F(x)$. However, Bézout's theorem gives the upper bound of the number of solutions. In order to evaluate an exact number of solutions, some linear reduction method or the nonlinear reduction method can be used to reduce the target system and give a smaller E [71].

In the second stage, as t moves from 0 to 1, numerical continuation methods trace the paths that originate at the solutions of the start system toward the solutions of the target system, and the solving process is shown in Fig. 4. The target system $F(x)$ can be solved based on the known and smooth solutions of $G(x)$, where the nonsingular isolated solutions are the required solutions.

During the past several decades, many applications have adopted the polynomial homotopy algorithm (PHA) to solve engineering problems, and the PHCpack was developed by Dr. Jan Verschelde to solve the polynomial system [121]. The construction of the start system and the homotopy mapping equation, the numerical continuation, or path-following algorithms are all included in this package. Therefore, it is very user-friendly for us to solve the SHE equations.

VI. Real-Time Implementation of Algebraic Methods

As the SHE equations are challenging to solve by algebraic algorithms in real time, the most common way to implement these algebraic algorithms is still on offline and storing the results in a lookup table. However, the lookup table method costs lots of on-chip memories, and the accuracy of the output switching angles is limited due to the sampled modulation indexes. To overcome this problem, two aforementioned algebraic algorithms were proposed to be implemented in real time.

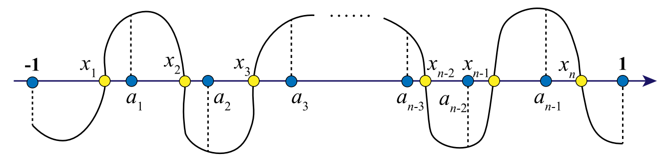

Fig. 5. Solve n nonoverlapping intervals by Sturm's theorem.

A. Algebra-Numerical Hybrid Algorithm

In [64], an algebra-numerical hybrid algorithm is used to realize SHE in real time, in which the solving process is divided into the offline stage and the online stage. In the offline stage, the SHE equations are simplified by using the elementary symmetric polynomials at first. Then, the Gröbner basis with undetermined parameters of the reduced polynomial system (7) is computed so that (7) can be converted to a group of univariate linear equations. Taking the single-phase SHE equations with four switching angles as an example, the Gröbner basis with undetermined parameters has been described as follows, which needs to be stored in the microcontroller:

$$\begin{cases} e_1 - m = 0 \\ b_1 e_2 + c_1 = 0 \\ b_2 e_3 + c_2 = 0 \\ b_3 e_4 + c_3 = 0 \end{cases} \tag{32}$$

where

$$b_1 = 448m^4 - 1680m^2 + 1260$$

$$c_1 = -192m^6 + 1008m^4 - 1680m^2 + 945$$

$$b_2 = 672m^4 - 2520m^2 + 1890$$

$$c_2 = -64m^7 + 504m^5 - 1260m^3 + 945m$$

$$b_3 = 26880m^4 - 100800m^2 + 75600$$

$$c_3 = -256m^8 + 2880m^6 - 10080m^4 + 12600m^2 - 4725.$$

In the online stage, e_i in (32) can be directly solved according to the input modulation index m, so that the univariate (9) can be obtained. Therefore, the problem is converted to how to solve (9) in real time. Yang et al. [64] proposed to use Sturm's theorem to determine the intervals of distinct real roots of (9) and solve these real roots by using the bisection method. Sturm's theorem can divide n real roots into n nonoverlapping intervals, as shown in Fig. 5, which is completed by constructing the following Sturm chain:

$$\begin{aligned} & s_0(x) = f(x) \neq 0, s_1(x) = f'(x) \neq 0 \\ & s_2(x) = -rem(s_0(x), s_1(x)) \neq 0 \\ & \cdots \\ & s_n(x) = -rem(s_{n-2}(x), s_{n-1}(x)) \neq 0 \\ & s_{n+1}(x) = -rem(s_{n-1}(x), s_n(x)) = 0 \end{aligned} \tag{33}$$

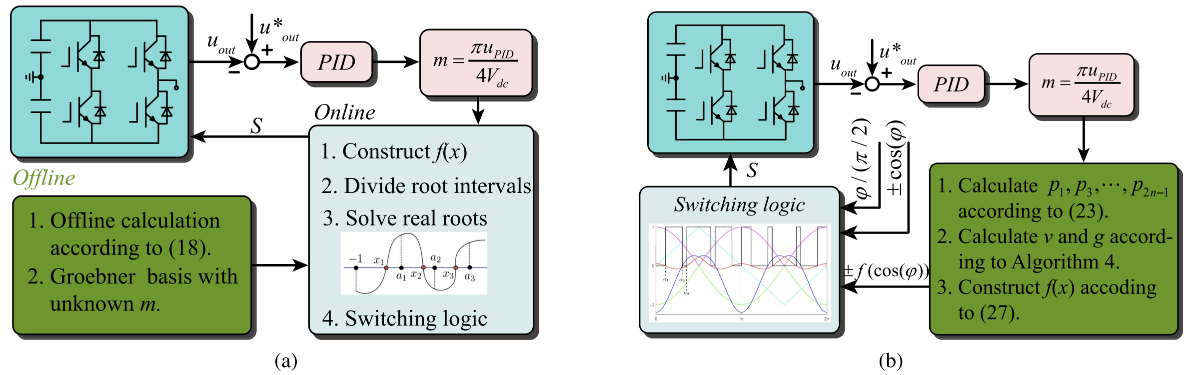

Fig. 6. Overall control system based on the two real-time implementation algorithms. (a) Algebra-numerical hybrid algorithm. (b) Chudnovsky algorithm real-time implementation.

where rem represents the remainder of the long division of two polynomials. Suppose any interval (a_1, a_2) and $f(a_1)f(a_2) \neq 0$; then, the number of sign change of (33) in a_1 and a_2 is $t(a_1)$ and $t(a_2)$, respectively. Sturm's theorem indicates that $f(x)$ has $t(a_1) - t(a_2)$ real roots within (a_1, a_2). The process of dividing the intervals is to find intervals (a_i, a_{i+1}) until $t(a_i) - t(a_{i+1}) = 1$, which means there is one real root in (a_i, a_{i+1}). After finite times division of intervals, the n real roots of the univariate equation can be isolated into n nonoverlapping intervals like Fig. 5. Finally, all n roots can quickly be found in these determined intervals using the bisection method or other iterative methods. This method has been implemented in an ARM Cortex-M4-based STM32F407 to control the multilevel inverters, and the execution time is around 0.3 ms, satisfying the requirement of the control system.

B. Chudnovsky Algorithm Real-Time Implementation

In [120], the Chudnovsky algorithm was proposed to control the two-level inverters in real time. As the above section described, the Chudnovsky algorithm only involves some conversions of mathematical formulas. Like the real-time method proposed in [64], the Chudnovsky algorithm is also to transform the SHE equations into a univariate high-order equation (9). The coefficients of $f(x)$ can be quickly solved using the deduced formulas of the Chudnovsky algorithm. Then, the roots of the univariate high-order equation $f(x)$ can be obtained by using a technique similar to Descartes' sign rule. This method redefines $f(x)$ with a substitution of $x = \cos(\varphi)$ and $x = -\cos(\varphi)$, where φ is the reference voltage angles, which is a period symmetric function within $0 \sim 2\pi$. Therefore, when $f(\cos(\varphi))$ equals zero, a root of the polynomial is detected. To obtain the switching command in real time, a technique similar to Descartes' sign rule is proposed, and the detailed analysis can refer to [120].

These two real-time implementation methods are used to control the single-phase inverters, and the overall control system for these two methods is shown in Fig. 6. The algebra-numerical hybrid can handle the three-phase inverters in theory because the Gröbner basis of the three-phase SHE equations with a given modulation index can be solved. However, when the modulation index is undetermined, the computational burden of solving the Gröbner basis significantly increases so that only the three-phase SHE equation with few switching angles can be solved by the algebra-numerical hybrid method. The Chudnovsky algorithm, due to the limitation of the principle, cannot be used to solve the three-phase SHE equations no matter online or offline. Therefore, these two real-time implementation methods are both used to solve the single-phase SHE problem. Nevertheless, the computing speed of the Chudnovsky algorithm is faster than the algebra-numerical hybrid method. The main reason is that there is an iterative process of the method proposed in [64] when using Sturm's theorem and numerical methods. Besides, limited by a huge computation burden of solving Gröbner basis, the solvable number of switching angles of the method proposed in [64] is much less than the Chudnovsky algorithm.

VII. Performance Comparison and Discussion

In the literature, each work claimed the superiority of their proposed method on various grounds and aspects. Thus, it is difficult to assess the true performance of any algorithm from the existing literature. Therefore, the performance of the algebraic algorithms mentioned above will be compared and discussed in this section. For a fair comparison of the performance of algebraic algorithms, the computer with a 2.2-GHz quad-core i7-2720QM CPU and 8-GB RAM is used to solve the switching angles by these algorithms. The resultant elimination method, the Wu method, the Gröbner-basis-based method, and the Chudnovsky algorithm are all implemented in the software $Maple21$. As all of these methods are based on complete mathematical theory, the accuracy of switching angles solved by these methods is equal to zero. Hence, the performance of these methods can be assessed based on their capability to solve switching angles, the computational complexity of algorithms, and the applicability to various SHE models.

A. Applicability for Different SHE Models

As summarized in Fig. 2, the mathematical model of SHE can be divided into different categories. According to the dc supply of inverters, SHE models can be divided into equal-

TABLE IV
APPLICABILITY AND COMPLEXITY OF ALGEBRAIC ALGORITHMS FOR DIFFERENT SHE MODELS

	Equal-level/Single-phase Quarter-symmetry		Equal-level/Three-phase Quarter-symmetry		Equal-level/Single-phase Half-symmetry		Unequal-level/Single-phase Quarter-symmetry	
Method[1]	Applicability	Complexity	Applicability	Complexity	Applicability	Complexity	Applicability	Complexity
ESP	✓	Medium	✓	Medium	✗	/	✗	/
NI	✓	Easy	✓	Easy	✗	/	✗	/
REA	✓	Medium	✓	Medium	✓ (in theory)	Hard	✓ (in theory)	Hard
WA	✓	Medium	✓	Medium	✓ (in theory)	Hard	✓ (in theory)	Hard
GA	✓	Medium	✓	Medium	✓	Hard	✓	Hard
CA	✓	Easy	✗	/	✗	/	✗	/
PHA	✓	Medium	✓	Medium	✓	Medium	✓	Medium

ESP: elementary symmetric polynomials; NI: Newton's identities; REA: resultant elimination algorithm; WA: Wu algorithm; GA: Gröbner basis algorithm; CA: Chudnovsky algorithm; PHA: polynomial homotopy algorithm.

and unequal-level models. According to the output waveform of inverters, SHE models can be divided into quarter-symmetry and half-symmetry. Besides, the difference between single- and three-phase systems leads to whether there are triple harmonics in the SHE equations. Based on the divergence of these mathematical models, the complexity of solving corresponding SHE equations is quite different. Therefore, the solving ability of these mentioned algebraic methods is related to the mathematical models. The applicability of these algebraic algorithms for different SHE mathematical models has been summarized in Table IV. It can be seen from Table IV that both the two simplification methods, the elementary-symmetric-polynomial-based method (ESP), and the Newton-identity-based method (NI), are constrained to handle only the quarter-symmetry SHE models. Besides, the resultant elimination algorithm (REA), Wu's algorithm (WA), and the Gröbner basis algorithm (GA) have the same properties that excels in solving quarter-symmetry models. With the half-symmetry models and the unequal-level models lacking symmetrical property, they cannot be simplified by ESP and NI. Although REA, WA, and GA can theoretically solve the half-symmetry models and unequal-level models, the number of solvable switching angles is notably limited. For example, GA exhibits higher computational capacity compared to REA and WA. However, up to a maximum of three switching angles can be solved using GA for the equal-level half-symmetry SHE models. Regarding the Chudnovsky algorithm, while it proves effective in solving single-phase SHE models with relative ease, its applicability is limited when dealing with other types of models. The PHA does not directly solve the original SHE equations, whereas it constructs a start system and then approximates the original solutions by using the continuation or path-following methods. Therefore, the PHA has the strongest applicability among the above algorithms.

B. Computational Complexity

As the algebraic algorithms are based on complete mathematical theory, these methods can obtain all the exact solutions. However, the efficiency of the these methods is quite different. In this section, the computational complexity of the aforementioned algebraic algorithms will be compared and discussed mainly in the aspect of computing time and the solvable number of switching angles. To facilitate a comprehensive comparison, steps 2 and 3 mentioned in Section III, which involve the simplification process and the triangularization process, are compared and discussed.

TABLE V
TIME COMPARISON BETWEEN THE TWO SIMPLIFICATION METHODS

	Switching points	$t_{\mathrm{ESP}}(s)$	$t_{\mathrm{NI}}(s)$
Single-phase	$N=4$	0.016	0.109
	$N=5$	0.063	0.156
	$N=6$	0.922	0.312
	$N=7$	19.234	0.453
	$N=8$	443.125	1.25
Three-phase	$N=4$	0.312	0.137
	$N=5$	0.969	0.49
	$N=6$	60.782	0.689
	$N=7$	2062.385	2.624
	$N=8$	122503.216	3.559

First, in the simplification process, the comparison is presented between the elementary-symmetric-polynomial-based method and the Newton-identity-based method in terms of computation time and the number of solvable switching angles. Table V shows the comparison results of the two simplification methods, where t_{ESP} represents the computing time of the elementary symmetric polynomial based method and t_{NI} represents the computing time of the Newton-identity-based method. It can be seen that the execution time of the Newton-identity-based method is much less than elementary-symmetric-polynomial-based method. Actually, when the number of switching angles exceeds eight, the elementary-symmetric-polynomial-based method totally fails to give the final results due to the huge computing burden, but the Newton's identities can solve more switching angles. Besides, it can be seen that the single-phase SHE equations are quite easier to solve than the three-phase SHE equations.

Then, in the triangularization process, the resultant elimination method, the Wu method, and the Gröbner basis method are used to solve the quarter-symmetry SHE models. Table VI shows the comparison results of the execution time and the solvable number of switching angles, where t_{RE}, t_{Wu}, and t_{GB} represent the execution time of the resultant elimination method, the Wu method, and the Gröbner-basis-based method, respectively. It can be seen that the efficiency of the Gröbner-basis-based

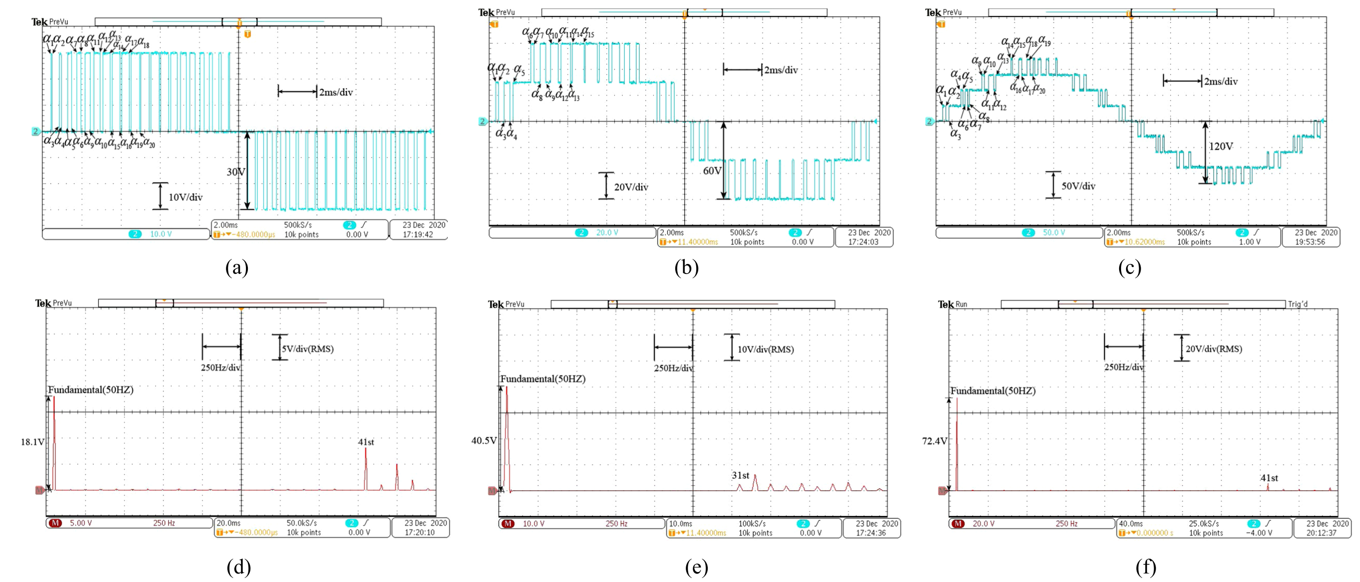

Fig. 7. Experimental results of the single-phase CHB. (a) Three-level phase voltage of the CHB with 20 switching angles. (b) Three-level phase voltage of the CHB with 15 switching angles. (c) Three-level phase voltage of the CHB with 20 switching angles. (d) FFT result of the output voltage (a). (e) FFT result of the output voltage (b). (f) FFT result of the output voltage (c).

TABLE VI
COMPUTING TIME OF ALGEBRAIC ALGORITHMS

	Switching points	$t_{\mathrm{GB}}(s)$	$t_{\mathrm{Wu}}(s)$	$t_{\mathrm{RE}}(s)$
Single-phase	$N=4$	0.024	0.358	0.238
	$N=5$	0.026	N/A	N/A
	$N=6$	0.101	N/A	N/A
	$N=7$	2.245	N/A	N/A
	$N=8$	202.86	N/A	N/A
Three-phase	$N=3$	0.014	0.111	0.047
	$N=4$	0.042	5.415	N/A
	$N=5$	2.269	N/A	N/A

method is higher than that of the Wu method and the resultant elimination method. The Gröbner basis method offers solutions for eight switching angles in single-phase SHE equations and five switching angles in three-level SHE equations, whereas the resultant elimination method and the Wu method provide solutions for only three and four switching angles, respectively.

Besides, to compare the computational complexity of the whole solving process, the resultant elimination method, the Wu method, and the Gröbner basis method combined with the two simplification methods are compared in terms of the execution time and the solvable number of switching angles. In the Table VII, it can be seen that the efficiency of these algebraic algorithms has been improved compared with Table VI, and the solvable number of switching angles has been increased by using the simplification methods. Besides, the solving ability of these algorithms in the Gröbner-basis-based method is greater than that in the resultant elimination method and the Wu method. This is different from the results in Table VI that the solving ability of the Wu method is greater than the ability of the resultant elimination method. Therefore, by using the simplification methods, the resultant elimination method has more improved than the Wu method.

VIII. Experimental Results

As any algebraic algorithms can derive the exact solutions for SHE equations, the solutions obtained are the same regardless of the algorithm applied. Therefore, the experimental results of the SHEPWM solved by algebraic algorithms will remain uniform. This section chooses single-phase SHE equations with 15 and 20 switching angles and three-phase SHE equations with 12 switching angles as example to verify the correctness and effectiveness of algebraic algorithms. The experimental study is established on five-, seven-, and nine-level cascaded H-bridge inverters, in which the IRFP250N MOSFETs are used as switching devices, the ADum1400 is used as the isolator, and the STM32F407 are used as the controller to generate the SHEPWM driven signal. The dc power supply of each H-bridge is set to 30 V. The experiments on both single- and three-phase inverters have been carried out, and the results validate the effectiveness of the switching angles computed by the algebraic methods.

In the single-phase experiments, the results of switching angles are solved by the Chudnovsky algorithm proposed in [120]. As mentioned above, the Chudnovsky algorithm has the strongest ability to solve single-phase SHE equations. In the experimental case of single phase as shown in Fig. 7, 15 and 20 switching angles are randomly selected to be verified. As shown in the fast Fourier transform (FFT) results, for 20 switching angles, all the harmonics before 41st are precisely eliminated. Besides, for 15 switching angles, all the harmonics before 31st are eliminated very well.

In the three-phase experiments, the results of switching angles are solved by the composed algorithm based on the Newton-identity- and Gröbner-basis-based methods proposed in [62], which has the strongest ability to solve the three-phase SHE equations. Fig. 8 shows the experimental results of the three-phase SHE equations with 12 switching angles when the

TABLE VII
COMPUTING TIME OF THE RESULTANT ELIMINATION METHOD, THE WU METHOD, AND THE GRÖBNER-BASIS-BASED METHOD WITH SIMPLIFICATION METHOD FOR QUARTER-SYMMETRY SHE EQUATIONS

		With symmetric polynomial			With Newton's identities		
	Switching points	$t_{\mathrm{GB}}(s)$	$t_{\mathrm{Wu}}(s)$	$t_{\mathrm{RE}}(s)$	$t_{\mathrm{GB}}(s)$	$t_{\mathrm{Wu}}(s)$	$t_{\mathrm{RE}}(s)$
Single-phase	$N = 4$	0.001	0.015	0.009	0.009	0.055	0.078
	$N = 5$	0.008	0.054	0.025	0.016	0.079	0.109
	$N = 6$	0.014	0.129	0.168	0.019	0.169	0.235
	$N = 7$	0.018	0.157	0.204	0.021	0.223	0.237
	$N = 8$	0.106	N/A	N/A	0.189	N/A	N/A
Three-phase	$N = 4$	0.007	0.187	0.016	0.141	0.25	0.031
	$N = 5$	0.047	3968.141	0.031	0.375	0.338	0.047
	$N = 6$	0.5	N/A	0.235	3.453	N/A	0.219
	$N = 7$	8.406	N/A	N/A	26.140	N/A	N/A
	$N = 8$	174.765	N/A	N/A	2119.5	N/A	N/A

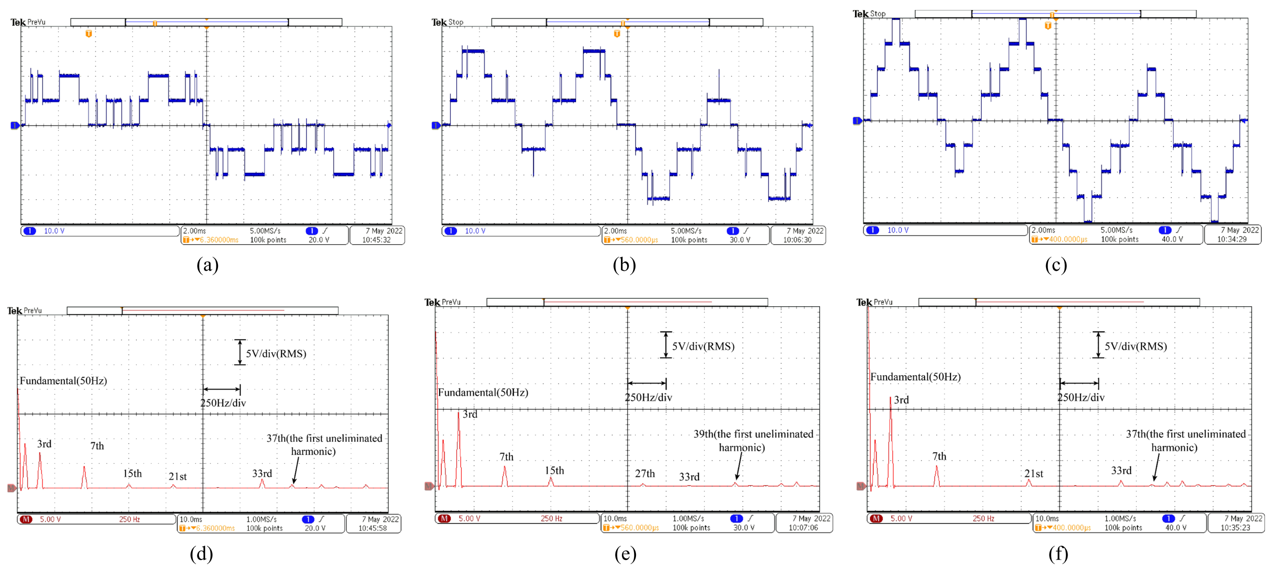

Fig. 8. Experimental results of the three-phase CHB with 12 switching angles under $m = 0.8$. (a) Three-level output phase voltage. (b) Seven-level output phase voltage. (c) Nine-level output phase voltage. (d) FFT result of the output voltage (a). (e) FFT result of the output voltage (b). (f) FFT result of the output voltage (c).

modulation index is 0.8. Since multiple solutions can be solved by algebraic algorithms, three groups' solutions are randomly selected to verify the correctness of the reviewed methods. It can be seen that the harmonics before 37th (except for triple harmonics) have been precisely eliminated.

IX. CONCLUSION

The SHE technique is an important modulation method aiming to solve the harmonic elimination problem for converters. The biggest challenge for using the SHE technique is to solve the nonlinear and transcendental SHE equations. Algebraic algorithms are powerful methods in solving SHE equations, which can find all exact solutions without requirements on choosing initial values. This review article comprehensive explored the existing algebraic algorithms, including the principle and the solving process of these methods. In this article, two simplification methods for reducing the degree of SHE equations and five algebraic methods for solving the SHE equations are introduced. Besides, the efficiency improvement of the algebraic method and the real-time implementation of SHE are also contained in this article. Finally, a detailed comparison and evaluation of these algebraic algorithms are given, and the experimental results verify the correctness of the solutions obtained by algebraic algorithms.

In the whole solving procedure of algebraic algorithms, the performance of simplification methods and algebraic methods determines the solving efficiency and ability for SHE equations. At present, the development of algebraic algorithms is relatively complete and mature. However, some future works are still reserve to be promoted.

1) *Simplification methods:* The existing simplification methods are all based on the principle of symmetric polynomial, which requires that SHE equations must be QW symmetry. No research has ever discussed whether there are other ways to equivalent transform SHE equations into a reduced polynomials system or not. Therefore, further research still need to be explored in the simplification methods.
2) *Efficiency improvement of algebraic methods:* The most algebraic methods, such as the Wu method and the Gröbner basis-based method, are all based on the universal

algorithm packed in software, which may limit the best performance of these methods. To improve the efficiency of solving SHE equations, further attempts can be made to design the targeted algorithms for SHE equations and get rid of redundant procedures of the universal algorithms.

3) *Real-time implementation:* Although there are two real-time methods that can be implemented by algebraic algorithms, they have only been applied to solve single-phase SHE equations. However, most industrial applications are three-phase systems, such as motor drive and grid-connected converters. Based on the existing real-time methods, the Chudnovsky algorithm cannot solve three-phase SHE equations because of the limitation of the principle; however, the hybrid method proposed in [64] is limited by the computational burden, which may be improved by the promotion of the above two points. Therefore, further work can be conducted on this aspect.

4) *Optimal SHE switching pattern:* Algebraic algorithms can solve all the possible switching patterns for SHE, which have been discussed in many literature works [62], [81], [102], [103], [104]. However, there is the absence of discussion about how to utilize these switching patterns. Therefore, a generalized method to obtain the optimal switching pattern can be further conducted based on the requirements of industrial applications, such as the conducting time of each switching tube, the value of total harmonic distortion, and the balance of capacitor voltage.

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Chenxu Wang received the B.Eng. degree in automation from the Qingdao University of Technology, Qingdao, China, in 2017. She is currently working toward the Ph.D. degree in electrical engineering with the School of Mechanical and Electrical Engineering, China University of Mining and Technology (Beijing), Beijing, China.

Her main research interests include modeling and control of high-power converters, model-predictive control of power electronics converters, selective harmonic elimination, and the applications of computer algebra in power electronics.

Qi Zhang (Student Member, IEEE) received the B.Eng. and M.Sc. degrees in electrical engineering from the China University of Mining and Technology (Beijing), Beijing, China, in 2015 and 2018, respectively, and the Ph.D. degree in energy technology from Aalborg University, Aalborg, Denmark, in 2023.

He is currently a Postdoctoral Researcher with AAU Energy, Aalborg University. His main research interests include modeling, control, fault analysis, and transient stability of power-electronic-dominated power systems; control of power electronic converters; selective harmonic elimination; HVDC transmission system; and HVDC tapping technology.

Wensheng Yu received the Ph.D. degree in dynamics and control from Peking University, Beijing, China, in 1998.

In 1998, he joined the Institute of Automation, Chinese Academy of Sciences, Beijing, where he became a Professor in 2004. He was a Visiting Research Fellow with the University of Melbourne, Melbourne, VIC, Australia, from 2001 to 2002. He was a Professor with the Shanghai Key Laboratory of Trustworthy Computing, East China Normal University, Shanghai, China, from 2009 to 2014. He is currently a Professor with the School of Electronic Engineering, Beijing University of Posts and Telecommunications, Beijing. He has authored or coauthored numerous articles in journals and conference proceedings, which span the fundamental fields of control theory and control engineering and applied mathematics. His current research interests include robust and optimal control, adaptive filter theory and linear estimation, control theory and control engineering, robotics and control, complex systems and control, and signal processing and communications.

Kehu Yang (Member, IEEE) received the B.S. degree in electrical engineering from Northwestern Polytechnical University, Xi’an, China, in 2003, and the Ph.D. degree in control theory and control engineering from the Institute of Automation, Chinese Academy of Sciences, Beijing, China, in 2009.

From 2013 to 2014, he was a Postdoctoral Research Fellow with the Department of Electrical and Computer Engineering, The Ohio State University, Columbus, OH, USA, working on selective harmonic elimination technology and its applications in high-power converter and electric drives. He is currently a Professor with the China University of Mining and Technology (Beijing), Beijing, and also with the Key Laboratory of Intelligent Mining and Robotics, Ministry of Emergency Management of the People’s Republic of China, Inner Mongolia Research Institute of China University of Mining and Technology (Beijing), Ordos, China. His research interests include harmonic control technologies and modeling and control methods for high-power converters.