# Letters

# Optimal Charging of Nonlinear Capacitors

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***Abstract*—Constant-current charging is the optimal solution for charging linear (fixed) capacitors. In this letter, we extend this principle to nonlinear capacitors using a variational method. We address the case where the capacitance depends only on the applied voltage. We show that a nonlinear capacitor stores energy electrostatically and by another mean, depending on the phenomena behind the variation of the capacitance. We compute the optimal charging voltage curves for a linearly increasing or decreasing capacitance with the bias voltage. We highlight that the efficiency of the charging is improved if the capacitance increases with the applied voltage, and vice versa. We finally apply our results to two types of nonlinear capacitors that are widely used in power electronics: Supercapacitors and class II ceramic capacitors. We demonstrate that the charging efficiency is higher for supercapacitors than for ceramic capacitors.**

***Index Terms*—Capacitors, capacitive energy storage, nonlinear circuits, optimization methods.**

## I. Introduction

CHARGING a capacitor with a step function from zero to the supply voltage $V_{dd}$ dissipates $\frac{1}{2}CV_{dd}^2$ by Joule heating in the access resistor $R$. As the capacitor stores the same amount of electrostatic energy, the process has an efficiency of 50%. Fortunately, the efficiency can be improved by avoiding the current peak at the beginning of the charging. For example, if the capacitor is charged in a time $T$ with a constant current, the Joule heating becomes $\frac{RC}{T}CV_{dd}^2$ [1]. For a leakage-free capacitor, the higher the ramp time $T$, the lower the dissipation. For that reason, constant-current charging is also called "adiabatic charging."

The principle of adiabatic charging is well established for linear (fixed) capacitors. However, in some cases, such as supercapacitors (SCs), ceramic capacitors (CCs), varicaps, or MOSFET gate capacitance, the capacitance varies significantly with the bias voltage. The case of SCs is relevant. Their capacitance increases about 50% over their operating voltage [2]–[5]. As SCs are used for energy storage, the impact of this variation on the efficiency of a charge/discharge cycle is a relevant question. Class II CCs have the opposite behavior. Their capacitance decreases significantly with the applied voltage [6]. For the real CCs depicted in Table I, the drop is in the order of 60% of the primary capacitance. This drop causes distortion [7] and may affect the operation of CCs-based power conversion systems.

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| Ref. | type | $C_0$ | $V_0$ | $V_{dd}$ | $\frac{V_0}{V_{dd}}$ | $\xi$ |
|---|---|---|---|---|---|---|
| [15] | MLCC | 10 $\mu$F | 79 V | 50 V | 1.59 | 0.685 |
| [6] | Disk CC | 13 nF | 1.6 kV | 1.0 kV | 1.62 | 0.700 |
| [16] | MLCC | 2.1 nF | 400 V | 200 V | 2.0 | 0.770 |
| Lin. cap. | - | - | - | - | - | 0.833 |
| [2] | SC | 1.9 kF | 4.5 V | 2.6 V | 1.73 | 0.849 |
| [3] | SC | 6.74 F | 147 V | 100 V | 1.47 | 0.850 |
| [4] | SC | 1.98 kF | 2.94 V | 2.55 V | 1.15 | 0.852 |

Charging a nonlinear capacitor through a resistor [1], [8], [9] or a linear inductor [10] is an ongoing question in electronics. Some studies mentioned that constant-current charging of nonlinear capacitors optimizes the efficiency [1], [9]. This principle is established for a few particular cases but has never been generalized to any kind of nonlinear capacitor. In this letter, we prove by the same variational approach as Wang [11] that constant-current charging is the optimal solution for any $C(V)$ function. We then address the cases of a linearly increasing or decreasing capacitance with the applied voltage.

## II. General Case

The capacitance $C$ is defined as the ratio between the stored charge $q$ and the bias $V$

$$C(V) = \frac{q}{V}. \tag{1}$$

When an electric current $i$ flows through the capacitor, the charge $q$ evolves according to

$$i = \frac{\mathrm{d}q}{\mathrm{d}t} = \dot{q}. \tag{2}$$

Combining (1) with (2) leads to

$$i = \dot{C}(V)V + \dot{V}C(V). \tag{3}$$

The first term $\dot{C}(V)V$ vanishes only if the capacitor is linear. In the general case, we assume the capacitance is a continuous and differentiable function of the voltage. Thus, $\dot{C}(V) \neq 0$ and both terms of (3) must be taken into account. For the sake of clarity, in the following paragraphs, the capacitance and its time derivative are denoted as $C$ and $\dot{C}$, respectively, even though they depend on the voltage.

### A. Finding the Optimal Current Function

We assume leak conductances in parallel to the capacitor are negligible. If the capacitor is charged via a fixed resistor $R$, the Joule heating in $R$ is written as

$$E_J(t) = \int_0^t Ri^2 \mathrm{d}t. \tag{4}$$

The resistor $R$ stands for the equivalent series resistance of the whole system, including the resistance of the capacitor. In order to maximize the charging efficiency, we seek a current function $i$ that minimize (4). According to the Euler–Lagrange equation, $i$ must satisfy

$$\frac{\partial i^2}{\partial V} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial i^2}{\partial \dot{V}}\right) = 0. \tag{5}$$

The two partial derivatives in (5) are calculated from the expression of the current (3)

$$\begin{cases} \dfrac{\partial i^2}{\partial V} = 2\dot{C}^2 V + 2\dot{C}C\dot{V} \\ \dfrac{\partial i^2}{\partial \dot{V}} = 2\dot{C}CV + 2\dot{V}C^2 = f(C, \dot{C}, V, \dot{V}) \end{cases}. \tag{6}$$

In order to calculate the second term of (5), we must differentiate the function $f$

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial \dot{C}}\frac{\mathrm{d}\dot{C}}{\mathrm{d}t} + \frac{\partial f}{\partial C}\frac{\mathrm{d}C}{\mathrm{d}t} + \frac{\partial f}{\partial \dot{V}}\frac{\mathrm{d}\dot{V}}{\mathrm{d}t} + \frac{\partial f}{\partial V}\frac{\mathrm{d}V}{\mathrm{d}t}. \tag{7}$$

In (7), substituting $f$ by its expression leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial i^2}{\partial \dot{V}}\right) = 2CV\ddot{C} + (2\dot{C}V + 4C\dot{V})\dot{C} + 2C^2\ddot{V} + 2\dot{C}C\dot{V}. \tag{8}$$

After subtracting (6) and (8), the Euler–Lagrange equation (5) becomes

$$C\ddot{V} + 2\dot{V}\dot{C} + \ddot{C}V = 0. \tag{9}$$

This expression matches the time derivative of (3). Thus, the Joule heating is minimum when

$$\frac{\mathrm{d}i}{\mathrm{d}t} = 0. \tag{10}$$

Therefore, constant-current charging is the most efficient way of charging a nonlinear capacitor. This assertion is true for any $C(V)$ function, thus generalizing the result of [1]. Interestingly, many studies on SCs already use constant-current charging [3], [4], [12], [13].

If the $C(V)$ curve is anhysteretic, the discharge is the time-reversal symmetry of the charge. The optimal current $i_0$ is reversed during the discharge but keeps the same absolute value and causes the same Joule heating. For this reason, we only focus on the charging phase.

### B. Energy Distribution in the Capacitor

We now address the question of how the energy is distributed in the capacitor. We assume the function $C(V)$ is not affected by other parameters, such as temperature. Raising the charge of the capacitor from $q$ to $q + \mathrm{d}q$ needs an external work $\mathrm{d}E = \mathrm{d}qV$. The charge $\mathrm{d}q$ is calculated by differentiating (1). The elementary work $\mathrm{d}E$ becomes

$$\mathrm{d}E = \left(C + V\frac{\partial C}{\partial V}\right) V\mathrm{d}V. \tag{11}$$

Starting from a discharged capacitor, the energy stored in the capacitor $E$ is given by the integration of (11) over $V$

$$E = \int_0^V \left(C + V\frac{\partial C}{\partial V}\right) V\mathrm{d}V.$$

Integration by parts leads to $E = E_{\text{el}} + E_{\text{ad}}$ with

$$E_{\text{el}} = \frac{1}{2}CV^2 \quad E_{\text{ad}} = \frac{1}{2}\int_0^V \frac{\partial C}{\partial V}V^2\mathrm{d}V. \tag{12}$$

The energy $E$ arises from two contributions: $E_{\text{el}}$ is the usual electrostatic energy, i.e., the energy of the electric field in the capacitor. The second term $E_{\text{ad}}$ is an additional amount of energy. In the general case, the nature of $E_{\text{ad}}$ (mechanical, electrostatic, chemical, etc.) cannot be specified, since it depends on the phenomenon that changes the capacitance. As an example, in an MEMS varactor $E_{\text{ad}}$ is the mechanical energy stored by the device. If no dissipative forces act in the mechanism that shifts the capacitance, $E_{\text{ad}}$ is fully recovered during the discharge.

Assuming $C(V)$ is monotonic over the range $[0 : V_{dd}]$, three cases can be discussed, which are as follows.

1) $\frac{\partial C}{\partial V} > 0$: The capacitance increases with the bias voltage, so $E_{\text{ad}}$ is positive. Thus, at the voltage $V_{dd}$ the nonlinear capacitor stores more energy than a fixed one with a capacitance $C(V_{dd})$.
2) $\frac{\partial C}{\partial V} = 0$: The capacitor is linear so $E_{\text{ad}} = 0$.
3) $\frac{\partial C}{\partial V} < 0$: The capacitance decreases with the bias voltage, so $E_{\text{ad}}$ is negative. Thus, at the voltage $V_{dd}$ the nonlinear capacitor stores less energy than a linear one with a capacitance $C(V_{dd})$. The capacitor provides the additional energy from an internal source.

### C. Joule Heating in the Resistor

If the capacitor is charged from zero to the final voltage $V_{dd}$ in a time $T$, the charging current $i_0$ is written as

$$i_0 = \frac{C(V_{dd})V_{dd}}{T}. \tag{13}$$

The Joule heating [cf. (4)] becomes

$$E_J = \frac{RC(V_{dd})}{T}C(V_{dd})V_{dd}^2. \tag{14}$$

This result is the same as the Joule heating encountered during the adiabatic charging of a linear capacitor with a capacitance $C(V_{dd})$ [1].

### D. Optimal Voltage Curve

The simplest way of charging a nonlinear capacitor is to use a constant current source. However, if only a voltage source is available, one must know the optimal $V(t)$ curve that goes from zero to $V_{dd}$ in a time $T$. To answer this question, we start from the discharged state $q(0) = 0$. Assuming a current $i_0$ flows

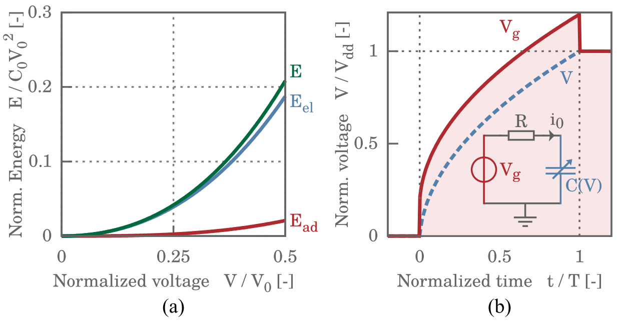


Fig. 1. (a) Electrostatic ($E_{\rm el}$), additional ($E_{\rm ad}$), and total ($E$) energies in the nonlinear capacitor versus the bias voltage. The capacitance is given by (18). (b) Optimal charging of the linearly increasing capacitor for $V_0 = 0.1V_{dd}$ and $R = 0.2\frac{V_{dd}}{i_0}$. The full red line depicts the generator voltage $V_g$, whereas the blue dashed line depicts the voltage $V$ across the capacitor.

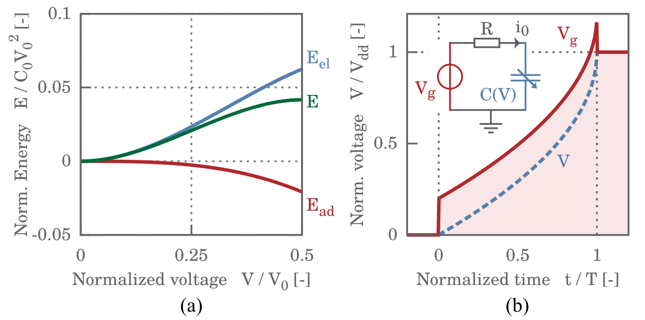


Fig. 2. (a) Electrostatic ($E_{\rm el}$), additional ($E_{\rm ad}$), and total ($E$) energies in the nonlinear capacitor versus the bias voltage. The capacitance is given by (21). (b) Optimal charging of the linearly decreasing capacitor, for $V_0 = 2V_{dd}$ and $R = 0.2\frac{V_{dd}}{i_0}$. The full red line depicts the generator voltage $V_g$, whereas the blue dashed line depicts the voltage $V$ across the capacitor.

through the capacitor, the charge $q(t)$ stored by the capacitor at the time $t$ is written as

$$q(t) = \int_0^t i(t)\mathrm{d}t = i_0 t. \tag{15}$$

Substituting $q$ by the definition of the capacitance (1) and $i_0$ by (13) leads to the equation of the optimal voltage curve

$$CV(t) - C(V_{dd})V_{dd}\frac{t}{T} = 0. \tag{16}$$

In order to follow the optimal voltage curve of (16), the generator must provide the following voltage

$$V_g(t) = V(t) + Ri_0. \tag{17}$$

The value of $R$ has an influence on the generator voltage $V_g(t)$ but not on the voltage across the capacitor $V(t)$. Knowing the optimal voltage curve $V_g(t)$ is valuable for applications where a nonlinear capacitor is charged by a voltage source, such as in regenerative braking with SCs [14]. In the following paragraphs, we focus on the cases of linearly increasing or decreasing nonlinear capacitors.

## III. Case of an Increasing Capacitance

In a general framework, we choose a linear $C(V)$ function that is able to fit the capacitance of a real device at least for small voltages

$$C(V) = C_0\left(1 + \frac{V}{V_0}\right) \tag{18}$$

where $C_0$ is the primary (unbiased) capacitance, and $V_0$ the voltage that double the capacitance. We emphasize that the following results apply qualitatively to any nonlinear capacitor with an increasing capacitance, such as SCs. The additional energy is calculated from (12) and (18)

$$E_{\rm ad} = \frac{C_0 V^3}{6V_0}. \tag{19}$$

As the capacitance increases with the bias voltage, $E_{\rm ad}$ is positive. The energy distribution in the capacitor is depicted in Fig. 1(a). At small bias voltages ($V \ll V_0$), $E_{\rm ad}$ is negligible against $E_{\rm el}$ so the nonlinear capacitor behaves like a fixed one. For large bias ($V \gg V_0$), the ratio between $E_{\rm el}$ and $E_{\rm ad}$ increases up to three. Therefore, the capacitor stores three-fourth of the energy electrostatically and one-fourth by another mean. These ratios arise from the linear relation between $C$ and $V$ (18).

Let us now calculate the optimal voltage curve for the linearly increasing capacitor. Injecting (18) in (16) leads to a second-order polynomial function, which has the following physical solution:

$$V(t) = \frac{1}{2}\left(\sqrt{V_0^2 + 4(V_{dd} + V_0)V_{dd}\frac{t}{T}} - V_0\right). \tag{20}$$

Fig. 1(b) depicts the optimal $V(t)$ curve and the corresponding generator voltage $V_g$ for $V_0 = 0.1V_{dd}$. The increase of $C$ causes the convex curvature of $V(t)$ measured experimentally on SCs [2], [4].

## IV. Case of a Decreasing Capacitance

We now address the case where the capacitance decreases with the bias voltage, as in CCs or in reverse-biased p-n junctions. We assume the following decreasing capacitance function:

$$C(V) = C_0\left(1 - \frac{V}{V_0}\right) \tag{21}$$

where $V_0$ is the voltage for which $C = 0$. As the capacitance must be positive, $V < V_0$. Fig. 2(a) depicts $E_{\rm ad}$, $E_{\rm el}$, and the total energy stored in the capacitor $E$ versus the bias voltage. The capacitor stores more electrostatic energy than the total energy stored (i.e., $E_{\rm el} > E$). This paradox is solved by considering the additional energy, as $C$ decreases with the applied voltage, $E_{\rm ad}$ is negative. Therefore, an energy intrinsic to the capacitor has been converted into electrostatic energy.

The optimal charging voltage curve is calculated by the same approach as in the case of the increasing capacitance. The most efficient $V(t)$ curve that goes from zero to the supply voltage $V_{dd}$ is given by

$$V(t) = \frac{1}{2}\left(V_0 - \sqrt{V_0^2 + 4(V_{dd} - V_0)V_{dd}\frac{t}{T}}\right). \tag{22}$$

Fig. 2(b) depicts the optimal $V(t)$ curve and the corresponding generator voltage $V_g$ for $V_0 = 2V_{dd}$. The decrease of $C$ during the charging causes the concave curvature of $V(t)$.

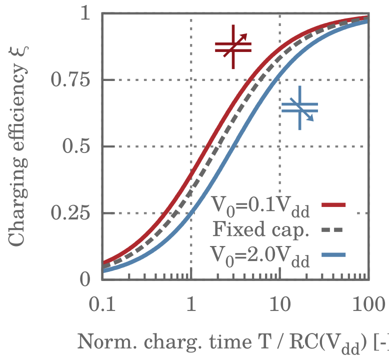


Fig. 3. Efficiency $\xi$ versus charging time $T$ for a linear capacitance (dashed line), an increasing capacitance ($V_0 = 0.1V_{dd}$, red plain line), and a decreasing capacitance ($V_0 = 2V_{dd}$, blue plain line). The charging time $T$ has been normalized by the time constant at the supply voltage $RC(V_{dd})$. This ensures that charging currents $i_0$ are the same when comparing the efficiency for different values of $V_0$.

## V. Efficiency of the Charging

We define the efficiency of the charging $\xi$ as the ratio between the energy stored at the voltage $V_{dd}$ and the energy provided by the generator

$$\xi = \frac{E}{E + E_J} = \frac{E_{\text{el}} + E_{\text{ad}}}{E_{\text{el}} + E_{\text{ad}} + E_J}. \quad (23)$$

Fig. 3 depicts the efficiency $\xi$ against the charging time $T$, for the following three cases.

1) A linear capacitor with a fixed capacitance.
2) A nonlinear capacitor with a linearly increasing capacitance for $V_0 = 0.1V_{dd}$. The efficiency is greater than for the linear capacitor; lower $V_0$, stronger the increase of $C$ and higher the efficiency.
3) A nonlinear capacitor with a linearly decreasing capacitance, for $V_0 = 2V_{dd}$. The efficiency is lower than for the linear capacitor; lower $V_0$, stronger the decay of $C$ and lower the efficiency.

The improvement of the efficiency results from the positive additional energy: If $\frac{\partial C}{\partial V} > 0$, more energy is transferred to the capacitor for the same Joule heating, and vice versa. In any case, $\xi$ increases with $T$ because the charging gets closer to an adiabatic process. For large charging times ($T \gg RC(V_{dd})$), the efficiency converges to one, since we assumed a leakage-free capacitor. With a parallel leakage resistor across the capacitor, $\xi$ would reach a maximum lower than one for a finite optimal charging time.

Table I depicts the efficiency calculated for experimental variable capacitors found in the literature. The parameters $C_0$ and $V_0$ were extracted from the experimental $C(V)$ curves, whereas $V_{dd}$ is the rated voltage of the capacitor. The ramp time $T$ is fixed to $T = 10RC(V_{dd})$. With this charging time, a linear capacitor is charged with an efficiency of $\xi = 0.833$. Table I lists that $\xi$ is reduced about 10%–20% with CCs, due to the decay of the capacitance during the charging. By contrast, $\xi$ increases about 2%–3% with SCs. Therefore, the variations of $C$ in real capacitors have a significant impact on the charging efficiency.

## VI. Conclusion

This letter demonstrates by a variational method that constant-current charging is the optimal way of charging a nonlinear capacitor. This principle is true for any continuous and differentiable capacitance function of the voltage, thus generalizing earlier studies.

During the charging, the variation of the capacitance leads to an additional term of energy. If the capacitance increases (decreases) with the applied voltage, the additional energy is positive (negative), the optimal charging voltage curve is convex (concave), and the efficiency is improved (reduced).

Applying these results to real SCs and CCs shows that the variation of capacitance impacts the efficiency up to 20%. These results must be taken into account when driving nonlinear capacitive charges, such as SCs or CCs. Furthermore, CCs store significantly less energy than expected from the simple calculation of the electrostatic energy. Less energy stored lowers the ability of the circuit to handle potential power spikes; this point must be considered when sizing decoupling capacitors.

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